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THE CLASS OF *BCC*-ALGEBRAS IS NOT A VARIETY

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Abstract. Kiyoshi Iséki posed an interesting problem whether the class of *BCK*-algebras is a variety. In connection with the problem, the author introduced a notion of *BCC*-algebras. In this note, we shall show that the class of *BCC*-algebras is not a variety. The result in this note was lectured on by Hiroakira Ono at Jagiellonian University in Poland in June 1981. A. Wroński [4], who began to study the problem since Ono's lecture, proved that the class of *BCK*-algebras is not a variety. (For the definitions and notations undefined here, see the reference [2].)

A *BCC-algebra* is an algebra $A = \langle A; \longrightarrow, 1\rangle$ of type $\langle 2, 0\rangle$ such that for every $x, y, z \in A$ the following conditions are satisfied:

(1) $(y \longrightarrow z) \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z)) = 1$,

(2) $x \longrightarrow x = 1$, (2) follows from the others.

(3) $x \longrightarrow 1 = 1$,

(4) $1 \longrightarrow x = x$,

(5) if $x \longrightarrow y = 1$ and $y \longrightarrow x = 1$, then $x = y$.

We have the axiom system of *BCK*-algebras (but dual form), if we exchange (1) for $(x \longrightarrow y) \longrightarrow ((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) = 1$.

Let F be the set of all terms, generated by two variabies x and y, in the *BCC*-language.

Definition 1. We define the set X, Y and $\mathbf{1}$ by the following inductive definition such that $\{X, Y, \mathbf{1}\}$ is a partition of F:

(1) $x \in X$, $y \in Y$ and $1 \in \mathbf{1}$,

(2) if $s \in X$ or $s \in Y$, then $s \longrightarrow t \in \mathbf{1}$,

(3) if $s \in \mathbf{1}$, then $s \longrightarrow t$ belongs to the set which t belongs to.

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Lemma 2. *If a class* K *of BCC-algebras is a variety, then there exist* $s \in X$ *and* $t \in Y$ *such that* $s = t \in Id(K)$.

Lemma 3. *The rightmost variable of any term in* X *(or* Y*) is* x *(or* y*, respectively).*

We now define Gentzen-type system LC. LC is a Gentzen's LJ-type system. In the following, Γ, Δ, Σ denote finite (possibly empty) sequences of terms separated by commas. Θ denotes a sequence which consists of at most one term. The followings are axioms and rules of inference of LC.

Axioms:

$$\alpha \Rightarrow \alpha \quad \text{(for any variavle } \alpha\text{)},$$

$$\Rightarrow 1.$$

Reles of Inference:

$$\text{cut: } \frac{\Gamma \Rightarrow t \quad \Sigma, t, \Delta \Rightarrow \Theta}{\Sigma, \Gamma, \Delta \Rightarrow \Theta}$$

$$\Rightarrow\longrightarrow: \frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \longrightarrow t} \qquad \longrightarrow\Rightarrow: \frac{\Gamma \Rightarrow s \quad \Sigma, t, \Delta \Rightarrow \Theta}{\Sigma, \Gamma, s \longrightarrow t, \Delta \Rightarrow \Theta}$$

$$T\Rightarrow: \frac{\Gamma, \Delta \Rightarrow \Theta}{\Gamma, t, \Delta \Rightarrow \Theta} \qquad \Rightarrow T: \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow t}.$$

Remark. For any term t, $t \Rightarrow t$ is provable in LC.

By the induction on the length of proof, we can prove that $s \longrightarrow t = 1$ is satisfied in all BCC-algebras if $s \Rightarrow t$ is provable in LC. Conversely, we can show that $s \Rightarrow t$ is provable in LC if $s \longrightarrow t = 1$ is satisfied in all BCC-algebras, by using the Lindenbaum algebra of LC. Hence, by the condition (5), we have

Theorem 4. *For any* s *and* t, $s = t$ *is satisfied in all BCC-algebras if and only if both* $s \Rightarrow t$ *and* $t \Rightarrow s$ *are provable in* LC.

Remark. Let LKK is a Gentzen-type system obtained form LC by adding to it the following rule of inference:

$$I\Rightarrow: \frac{\Gamma, s, t, \Delta \Rightarrow \Theta}{\Gamma, t, s, \Delta \Rightarrow \Theta}.$$

We can also show that $s = t$ is satisfied in all BCK-algebras if and only if both $s \Rightarrow t$

and $t \Rightarrow s$ are provable in LKK.

Similarly to Gentzen [1], we can prove the cut elimination theorem.

Theorem 5. (Cut Elimination Theorem). *If* $\Gamma \Rightarrow \Theta$ *is provable in* LC (*or* LKK), *then it is provable without a cut in* LC (*or* LKK, *respectively*).

Corollary 6. *Both the word problems for free BCC-algebra and for free BCK-algebra are solvable.*

Lemma 7. *If* Γ, $s \Rightarrow t$ *is provable in* LC, *then either the rightmost variable of* s *is the same as that of* t *or* $\Gamma \Rightarrow t$ *is provable in* LC.

Proof. We prove this by the induction on the length of the cut free proof of Γ, $s \Rightarrow t$. When the length is 1 (that is, the proof consists of only an axiom sequent), obviously this lemma holds.
We consider the case that $\longrightarrow \Rightarrow$ is the last rule of inference.

$$\longrightarrow \Rightarrow : \frac{\Gamma \Rightarrow s_1 \qquad \Sigma,\ s_2,\ \Delta \Rightarrow t}{\Sigma,\ \Gamma,\ s_1 \longrightarrow s_2,\ \Delta \Rightarrow t}\ .$$

If Δ is an empty sequence, then s equals $s_1 \longrightarrow s_2$. By the inductive hypothesis, either the rightmost variable of s_2 is the same as that of t or $\cancel{\Gamma} \Rightarrow t$ is provable in LC. Σ In the former case, the rightmost variable of s is the same as that of t. In the latter case, Σ, $\Gamma \Rightarrow t$ is provable in LC.

Suppose that Δ is not empty. We can denote Δ by Δ_1, s. By the inductive hypothesis, either the rightmost variable of s is the same as that of t or Σ, s_2, $\Delta_1 \Rightarrow t$ is provable in LC. In the former case, of course, the rightmost variable of s is the same as that of t. In the latter case, Σ, Γ, $s_1 \longrightarrow s_2$, $\Delta_1 \Rightarrow t$ is provable in LC.

In the case that another is the last rule, the proof is similar. Q. E. D.

Lemma 8. *If an identity* $s = t$ *is satisfied in all BCC-algebras and* $s \in X$, *then the rightmost variable of* s *is the same as that of* t.

Proof. By Theorem 4, $t \Rightarrow s$ is provable in LC. By Lemma 7, either the rightmost variable of s is the same as that of t or $\Rightarrow s$ is provable in LC. Suppose that $\Rightarrow s$ is provable in LC. Then both $1 \Rightarrow s$ and $s \Rightarrow 1$ are provable in LC. Therefore, $s = 1$ is satisfied in all *BCC*-algebras. But $s = 1$ is not satisfied in the simple *BCC*-algebra which has two elements. So $\Rightarrow s$ is not provable in LC. Q. E. D.

By Lemma 2, Lemma 3 and Lemma 8, we have

Theorem 9. *The class of all BCC-agebras is not a variety.*

References

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[2] G. Grätzer: Universal algebra (second edition). Springer-Verlag New York Inc., 1979.

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