The finite model property of the intermediate propositional logics on finite slices

By Yuichi KOMORI

(Communicated by S. Furuya)

There had been an interesting problem whether any intermediate propositional logic had the finite model property (fmp). Jankov [3] answered to this problem that there exists an intermediate logic without fmp. On the other hand, Hosoi [2] defined a classification of the intermediate logics by the notion of slice. Jankov's example of the logic without fmp is known to belong to the ω-th slice $\mathscr{S}_\omega$. It has remained open whether any intermediate logic on the finite slices has fmp.

In this paper, we will show that any intermediate logic on the finite slices has fmp. It should be mentioned here that Mr. Masahiko Sato has obtained the same result independently.

§ 1. Preliminaries.

By *a logic* we mean a set of propositional formulas which is closed under substitution and modus ponens and contains intuitionistic axioms. We write W and LJ (or LK) the set of all propositional formulas and the intuitionistic (or classical) propositional logic, respectively. Let P be a pseudo-Boolean algebra (PBA). We write $L(P)$ the set of all formulas valid in P. (For the definitions and notations undefined here, see the reference [2].)

We begin with the definition of *finite model property*.

DEFINITION 1.1. A logic L has *fmp* if there exists a set of finite PBAs $\{P_i \mid i \in I\}$ such that $L = \bigcap_{i \in I} L(P_i)$.

We write $\lambda(L)$ and $\lambda^n(L)$, respectively, the Lindenbaum algebra of a logic L and the subalgebra of $\lambda(L)$ generated by elements corresponding to propositional variables $a_1, a_2, \cdots, a_n$. We call an algebra P, *PBA* even for the case that P has only one element. Then, the following theorem is well-known.

THEOREM 1.2. *For any PBA P, $L(P)$ is a logic. Conversely, for any logic L, $L = L(\lambda(L)) = \bigcap_{n<\omega} L(\lambda^n(L))$.*

Now, we define a sequence of formulas A_n $(n=0, 1, 2, \cdots)$ which characterize the notion of slice.

DEFINITION 1.3.

$$\begin{cases} A_0 = a_0 \\ A_{n+1} = ((a_{n+1} \supset A_n) \supset a_{n+1}) \supset a_{n+1} \quad (n \geqq 0) . \end{cases}$$

The following theorem, which can be regarded as defining the notion of *slice*, was proved in [2].

THEOREM 1.4.

$L \in \mathscr{S}_0$ *if and only if* $A_0 \in L$

$L \in \mathscr{S}_n$ *if and only if* $A_n \in L$ *and* $A_{n-1} \notin L$ $(1 \leqq n < \omega)$

$L \in \mathscr{S}_\omega$ *if and only if for any* n $A_n \notin L$.

It is obvious that $\mathscr{S}_0 = \{W\}$ and $\mathscr{S}_1 = \{LK\}$. We write $\rightarrow$ the operation in PBAs corresponding to the logical connective $\supset$.

DEFINITION 1.5. Let P be a PBA and 1 be the maximum element of P. A subset F of P is *a filter* of P if it satisfies the following conditions:

(1) $1 \in F$.

(2) If $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

DEFINITION 1.6. Let P be a PBA and F be a filter of P. We define a relation $\sim_F$ on P as follows:

$$x \sim_F y \text{ if and only if } x \rightarrow y \in F \text{ and } y \rightarrow x \in F .$$

Then, the following theorem is well-known.

THEOREM 1.7. *For any PBA P and any filter F of P, the relation $\sim_F$ is an equivalence relation, and $P/\sim_F$ is naturally a PBA.* (We write P/F for $P/\sim_F$.)

DEFINITION 1.8. A PBA is *irreducible* if the set P^-, obtained from P by deleting the maximum element, has the maximum element. In what follows the maximum element of P^- is denoted by 2_P or 2.

Clearly, if P is an irreducible PBA, P^- is a PBA and isomorphic to $P/\{1, 2\}$.

LEMMA 1.9. *If P is an irreducible PBA and $A_n \in L(P)$ $(1 \leqq n < \omega)$, then $A_{n-1} \in L(P^-)$.*

PROOF. Suppose that $A_{n-1} \notin L(P^-)$. Then there exists an assignment f of P^- such that $f(A_{n-1}) \neq 1_{P^-}$. Then, we can prove that $f(a_i) < 1_{P^-}$ for any $i \leqq n-1$. Let f^* be an assignment of P such that $f^*(a_n) = 2_P$ and $f^*(a_i) = f(a_i)$ $(0 \leqq i \leqq n-1)$.

Then we can show $f^*(A_n)=2_P$. Hence $A_n \in L(P)$.

§ 2. Main result.

Though $\lambda^1(LJ)$ is infinite (cf. Nishimura [4]), we can prove the following lemma by Diego's technique [1].

LEMMA 2.1. *For any $m, n<\omega$, if $A_n \in L(P)$ and P has m generators, P is a finite PBA.*

PROOF. We prove this by the induction on n. We write $\#(P)$ the cardinality of P. If $n=0$, $L(P)=W$ and therefore $\#(P)=1$. Next, for any $x \in P$ such that $x \neq 1$, let S_x be the class of all maximal elements of the set $\{F \mid F$ is a filter of P and $x \notin F\}$. By Zorn's lemma, S_x is not empty. We put $V=\bigcup_{\substack{x\in P\\ x\neq 1}} S_x$ and $\Pi=(P/F)_{F\in V}$, where $(P/F)_{F\in V}$ is the Cartesian product of the family $\{P/F \mid F\in V\}$ of PBAs. Suppose that $h: P\to\Pi$ is a mapping such that $x\mapsto(h_F(x))_{F\in V}$, where $h_F: P\to P/F$ is the canonical mapping. Then h is a monomorphism. Hence, $\#(P)\leqq\#(\Pi)$. Thus the proof is completed if we prove that $\#(\Pi)$ is finite. This is ascertained by the following statements:

(1) $\exists q<\omega\,\forall F\in V \quad \#(P/F)\leqq q$.

(2) $\exists r<\omega \qquad \#(V)\leqq rq^m$.

PROOF OF (1). Let $x_1, x_2, \cdots, x_m$ be the generators of P. Then $\{h_F(x_1), h_F(x_2), \cdots, h_F(x_m)\}$ generates P/F. This PBA P/F is irreducible because F is a maximal filter. Since $A_n \in L(P/F)$, $A_{n-1}\in L((P/F)^-)$ by Lemma 1.9. By the inductive hypothesis, $k=\#(\lambda^m(LJ+A_{n-1}))$ is finite. We put $q=k+1$. Then, $\#((P/F)^-)\leqq q-1$ and $\#(P/F)\leqq q$.

PROOF OF (2). The cardinality r of the class of all PBAs whose cardinalities are at most q is finite if we identify isomorphic PBAs. It should be noted that the equality $F=F'$ does not follow from the property that P/F is isomorphic to P/F'. We consider the cardinality of the class $C=\{F' \mid P/F'$ is isomorphic to $P/F\}$. Suppose that $F\neq F'$. Then we can prove $h_F\neq\psi h_{F'}$, where ψ: $P/F'\to P/F$ is an isomorphism. Hence, $(h_F(x_1), \cdots, h_F(x_m))\neq(\psi h_{F'}(x_1), \cdots, \psi h_{F'}(x_m))$. Therefore $\#(C)\leqq q^m$. Hence, $\#(V)\leqq rq^m$.

Now, we can prove our main theorem.

THEOREM 2.2. *If $L\in\mathscr{S}_n$ for some $n<\omega$, then L has fmp.*

PROOF. By Theorem 1.2, $L=\bigcap_{n<\omega} L(\lambda^n(L))$, and by Theorem 1.4 and Lemma 2.1, $\lambda^n(L)$ is finite.

It is known that L has a Kripke model if L has *fmp* (Ono [5]). Hence, the following corollary is obvious.

COROLLARY 2.3. *If $L \in \mathscr{S}_n$ for some $n < \omega$, then L has a Kripke model.*

References

[1] Diego, A., Sur les algèbres de Hilbert, Collection de logique mathématique, Série A, No. 21, Paris, 1966.

[2] Hosoi, T., On intermediate logics I, J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 293-312.

[3] Jankov, V. A., Constructing a sequence of strongly independent superintuitionistic propositional calculi, Soviet Math. Dokl., **9** (1968), 806-807.

[4] Nishimura, I., On formulas of one variable in intuitionistic propositional calculus, J. Symbolic Logic, **25** (1960), 327-331.

[5] Ono, H., Kripke models and intermediate logics, Publ. RIMS, Kyoto Univ., **6** (1970), 461-476.

(Received December 21, 1973)

Department of Mathhmatics
Faculty of Science
Shizuoka University
Ohya Shizuoka
422, Japan