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A LAMBDA PROOF OF THE P-W THEOREM

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Abstract. The logical system P-W is an implicational non-commutative intuitionistic logic defined by axiom schemes $B = (b \to c) \to (a \to b) \to a \to c$. $B' = (a \to b) \to (b \to c) \to a \to c$. $I = a \to a$ with the rules of modus ponens and substitution. The P-W problem is a problem asking whether $\alpha = \beta$ holds if $\alpha \to \beta$ and $\beta \to \alpha$ are both provable in P-W. The answer is affirmative. The first to prove this was E. P. Martin by a semantical method.

In this paper. we give the first proof of Martin's theorem based on the theory of simply typed λ-calculus. This proof is obtained as a corollary to the main theorem of this paper. shown without using Martin's Theorem. that any closed hereditary right-maximal linear (HRML) λ-term of type $\alpha \to \alpha$ is $\beta\eta$-reducible to $\lambda x.x$. Here the HRML λ-terms correspond. via the Curry-Howard isomorphism. to the P-W proofs in natural deduction style.

§1. Introduction. The logical system P-W is an implicational non-commutative intuitionistic logic defined by axiom schemes

$$B = (b \to c) \to (a \to b) \to a \to c,$$
$$B' = (a \to b) \to (b \to c) \to a \to c,$$
$$I = a \to a,$$

with the rules of modus ponens and substitution. Martin's theorem states the following:

- *If $\alpha \to \beta$ and $\beta \to \alpha$ are provable in P-W, then $\alpha = \beta$.*

The truth of Martin's theorem had been an open problem, known as the "P-W problem" for more than twenty years, since Belnap had originally asked it. Finally E. P. Martin solved it affirmatively by showing the statement below using a semantical argument [8]. Powers showed the equivalence between this statement and the original one [10] by essentially obtaining the theory of combinators generated from B, B' and I.

- *No formula with form $\alpha \to \alpha$ is provable in P-W without using the axiom I.*

Martin's theorem means that the logic defined by B and B' is "progressive" in the sense that a deduction from an assumption α does not return to the start point itself. The deduction creates something new and does not yield the tautology $\alpha \to \alpha$, which would be a "logical nonsense"(cf. Fine and Martin [4]).

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Since Martin's solution was a semantical proof, finding a syntactic proof of the problem attracted some researchers. A syntactic proof in sequent calculus was obtained by the third author originally in June of 1993 ([9]) and the second author in August of 1993 ([6]). Also there have been several attempts to obtain a syntactic proof based on simply typed λ-calculus, where, by means of the Curry-Howard isomorphism, proofs are identified with λ-terms according to the "formulas-as-types" or "terms-as-proofs" correspondence.[1]

P. Trigg introduced, in [11], a notion of hereditary right-maximality for the combinators generated from B, B' and I. This notion accommodates a restricted exchange rule introduced by B and B', where we can exchange the position of the first two arguments but not the position of the third ones. This establishes, via the Curry-Howard isomorphism, the correspondence between proofs in the Hilbert-style formal system and the hereditary right-maximal $BB'I$-combinators.

The first author adapted this notion to linear λ-terms, where each variable contained in them occurs exactly once. This established, via the Curry-Howard isomorphism, the correspondence between P-W proofs in natural deduction style and hereditary right-maximal linear (HRML) λ-terms [5]. Furthermore, he showed in [5] that the statement below is equivalent to Martin's theorem.

- *Any HRML λ-term of type $\alpha \to \alpha$ is $\beta\eta$-reducible to $\lambda x.x$.*

The purpose of the present paper is to prove this theorem. Based on the transformations between P-W proofs in the Hilbert-style formal system and those in the natural deduction style system, we can define transformations between closed linear λ-terms and closed $BB'I$-combinators in both directions, whose composition is the identity up to $\beta\eta$-equality: a translation of a linear λ-term to a $BB'I$-combinator is obtained from an abstraction algorithm of basic combinators; in the other direction, a translation of a $BB'I$-combinator to a linear λ-term is obtained by the replacement of the combinators B, B' and I by the λ-terms $\lambda xyz.x(yz)$, $\lambda xyz.y(xz)$, and $\lambda x.x$. In the proof of the main theorem, we use two important properties of a β-reduction with respect to $\lambda xyz.x(yz)$ or $\lambda xyz.y(xz)$: namely, they preserve any subterm of form $P(QR)$, and never create a new η-redex.

In Section 2, we give definitions and theorems contained in [5], with some sketches of proofs, in order to make the paper self-contained.

In Section 3, we prove the main theorem (Theorem 3).

In Section 4, we derive, directly from the main theorem, Martin's theorem (Theorem 4) and the other statements due to Powers and Dwyer (Theorems 5, 6).

§2. Preliminaries. We consider B, B' and I as atomic combinators with the following reduction rules.

$$Bxyz \rhd x(yz)$$
$$B'xyz \rhd y(xz)$$
$$Ix \rhd x.$$

The set $\Lambda(BB'I)$ is defined as the smallest set containing every variable, the constants B, B' and I, closed under application and abstraction. Thus the terms in

[1] When R. Meyer visited Japan in 1991, he told the first author that he believed in a clear solution to the problem using λ-calculus.

$\Lambda(BB'I)$ will be mixed combinatory and λ-terms. We mean by a *term* an element in the set $\Lambda(BB'I)$.

The variables and the constants B, B' and I are said to be *atomic*. A term is *closed* iff it contains no free variables. A term is a *$BB'I$-term* iff it contains no λ's. A term is a *λ-term* iff it does not contain B, B' nor I. We denote the set of free variables in M by $\mathrm{FV}(M)$.

Definition 1 (Definition 2. Hirokawa [5]). *Let $x_0, x_1, \ldots, x_n$ be a sequence of distinct variables and M be a term. The* index *of M with respect to $x_0, x_1, \ldots, x_n$ is defined by*

$$\mathrm{idx}(M, x_0x_1\cdots x_n) = \begin{cases} \max\{i \mid 0 \le i \le n, x_i \in \mathrm{FV}(M)\} + 1 & \text{if } x_i \in \mathrm{FV}(M) \text{ for some } i = 0, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2 (Definition 3. Hirokawa [5]). *The set* $\mathrm{HRM}(x_0x_1\cdots x_n)$ *of* hereditary right-maximal *terms with respect to $x_0, x_1, \ldots, x_n$ is defined as follows.*

(1) *Every variable and the constants B, B', I are in* $\mathrm{HRM}(x_0x_1\cdots x_n)$.
(2) *If $M, N \in \mathrm{HRM}(x_0x_1\cdots x_n)$ and $\mathrm{idx}(M, x_0x_1\cdots x_n) \le \mathrm{idx}(N, x_0x_1\cdots x_n)$ then $(MN) \in \mathrm{HRM}(x_0x_1\cdots x_n)$.*
(3) *If $M \in \mathrm{HRM}(x_0x_1\cdots x_nx_{n+1})$ and $x_{n+1} \in \mathrm{FV}(M)$ then $\lambda x_{n+1}.M \in \mathrm{HRM}(x_0x_1\cdots x_n)$.*

Definition 3 (Definition 4. Hirokawa [5]). *A closed λ-term $\lambda x_0x_1\cdots x_n.M$ is said to be* hereditary right-maximal *iff $\mathrm{FV}(M) = \{x_0, x_1, \ldots, x_n\}$ and $M \in \mathrm{HRM}(x_0x_1\cdots x_n)$.*

We note that when $xM_1\cdots M_m \in \mathrm{HRM}(x_0x_1\cdots x_n)$, we have

$$\mathrm{idx}(x, x_0x_1\cdots x_m) \le \mathrm{idx}(M_1, x_0x_1\cdots x_m) \le \cdots \le \mathrm{idx}(M_m, x_0x_1\cdots x_m).$$

Definition 4 (Definition 5. Hirokawa [5]). *The set of* linear terms *is defined by induction thus*:

(1) *every variable and the constants B, B', I are linear,*
(2) *(MN) is linear if M, N are linear and $\mathrm{FV}(M) \cap \mathrm{FV}(N)$ is empty.*
(3) *$(\lambda x.M)$ is linear if M is linear and x occurs free in M.*

The set of closed hereditary right-maximal linear λ-terms is denoted by HRML.

Based on the transformations between the P-W proofs in the Hilbert-style system and those in the natural deduction style system, we can define transformations between closed linear λ-terms and closed $BB'I$-combinators in both directions, whose composition is the identity up to the $\beta\eta$-equality. Firstly, we define a transformation from a closed HRML linear λ-term M to a closed $BB'I$-combinator M^+.

Definition 5 (Definition 6. Hirokawa [5]). *Let $x_0, x_1, \ldots, x_n, x_{n+1}$ be a sequence of distinct variables ($n \ge -1$), M be a linear $BB'I$-term in $\mathrm{HRM}(x_0x_1\cdots x_nx_{n+1})$*

and $x_{n+1} \in \mathrm{FV}(M)$. *Then we define a linear term* $\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.M$ *as follows.*

$$\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.x_{n+1} = I$$
$$\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.Px_{n+1} = P$$
$$\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.PQ = \begin{cases} BP(\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.Q) & \text{if } \mathrm{idx}(P, x_0x_1\cdots x_n) \le \mathrm{idx}(Q, x_0x_1\cdots x_n) \\ B'(\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.Q)P & \text{if } \mathrm{idx}(P, x_0x_1\cdots x_n) > \mathrm{idx}(Q, x_0x_1\cdots x_n). \end{cases}$$

DEFINITION 6 (Definition 7. Hirokawa [5]). *Let* $x_0, x_1, \dots, x_n$ *be a sequence of distinct variables and* $X \in \Lambda(BB'I)$ *be a linear term in* $\mathrm{HRM}(x_0x_1\cdots x_n)$. *Then we define a linear* $BB'I$*-term* $(x_0x_1\cdots x_n; X)^*$ *in* $\mathrm{HRM}(x_0x_1\cdots x_n)$ *as follows.*

$$(x_0x_1\cdots x_n; X)^* = X \quad \text{if } X \text{ is atomic.}$$
$$(x_0x_1\cdots x_n; PQ)^* = (x_0x_1\cdots x_n; P)^*(x_0x_1\cdots x_n; Q)^*$$
$$(x_0x_1\cdots x_n; \lambda y.R)^* = \lambda_y^{x_0x_1\cdots x_n}.(x_0x_1\cdots x_n y; R)^*.$$

These definitions are well-defined [5]. Under the Curry-Howard isomorphism, a $BB'I$-term $(x_0x_1\cdots x_n; X)^*$ represents a Hilbert-style proof with types of $x_0, x_1, \dots, x_n$ as hypotheses ordered in this way. In particular, for a closed λ-term M, it can be shown that $\mathrm{FV}((;M)^*) = \mathrm{FV}(M) = \emptyset$, that is, $(;M)^*$ is a closed $BB'I$-term [5].

DEFINITION 7 (Definition 8. Hirokawa [5]). *Given a closed HRML* λ*-term* M, *we define* M^+ *as* $(;M)^*$.

Now we define a transformation from a closed $BB'I$-combinator M to a closed HRML linear λ-term $M^\diamond$:

DEFINITION 8 (Definition 9. Hirokawa [5]). *Given a term* $M \in \Lambda(BB'I)$, *a closed* λ*-term* $M^\diamond$ *is defined inductively by* $x^\diamond = x$, $B^\diamond = \lambda xyz.x(yz)$, $B'^\diamond = \lambda xyz.y(xz)$, $I^\diamond = \lambda x.x$, $(PQ)^\diamond = (P^\diamond Q^\diamond)$ *and* $(\lambda x.R)^\diamond = \lambda x.R^\diamond$.

THEOREM 1 (Theorem 2. Hirokawa [5]). *For a closed HRML* λ*-term* M, $M^{+\diamond} \overset{\beta\eta}{=} M$.

PROOF. We can prove $(x_0x_1\cdots x_n; M)^{*\diamond} \overset{\beta\eta}{=} M^\diamond$ by induction on M. By letting $n = -1$, we obtain the theorem. ⊣

THEOREM 2 (Theorem 3. Hirokawa [5]). *For a closed HRML* λ*-term* M *in* β*-normal form,* (1),(2) *and* (3) *are equivalent.*

(1) $M \overset{\eta}{\twoheadrightarrow} \lambda x.x$.
(2) M^+ *contains an* I.
(3) *The* η*-normal form of* M *does not contain a subterm of the form* $P(QR)$.

PROOF. (1) ⇒ (2): We can prove that for any term P and Q, if $P \overset{\eta}{\twoheadrightarrow} Q$ then $(x_0x_1\cdots x_n; P)^* = (x_0x_1\cdots x_n; Q)^*$, by induction on the length of the η-reduction. Thus $M \overset{\eta}{\twoheadrightarrow} \lambda x.x$ implies that $M^+ = (;M)^* = (;\lambda x.x)^* = \lambda_x.x = I$.

(2) ⇒ (1): Let $M = \lambda x_0\cdots x_n.xM_0\cdots M_m$ $(n \ge 0)$. Then

$$M^+ = \lambda_{x_0}\lambda_{x_1}^{x_0}\cdots\lambda_{x_n}^{x_0x_1\cdots x_{n-1}}.(x_0\cdots x_n; xM_0\cdots M_m)^*.$$

Now we can prove that for a β-normal form linear λ-term N in $\mathrm{HRM}(x_0x_1\cdots x_n)$ satisfying $\mathrm{FV}(N) \subseteq \{x_0, x_1, \dots, x_n\}$ that if N is not an abstraction, then $(x_0x_1\cdots x_n; N)^*$ does not contain an I, by induction on N. Hence $(x_0\cdots x_n; xM_0\cdots M_m)^*$

does not contain an I. Let i be the first i such that an I occurs as the result of the abstraction $\lambda_{x_i}^{x_0x_1\cdots x_{i-1}}.X$ where

$$X = \lambda_{x_{i+1}}^{x_0x_1\cdots x_i}\cdots\lambda_{x_n}^{x_0x_1\cdots x_{n-1}}.(x_0\cdots x_n; xM_0\cdots M_m)^*.$$

We can show that for a linear combinator X in $\mathrm{HRM}(x_0x_1\cdots x_nx_{n+1})$ not containing λ and I, if $x_{n+1}\in \mathrm{FV}(X)$ and $\lambda_{x_{n+1}}^{x_0x_1\cdots x_n}.X$ contains an I then $X = x_{n+1}$ by induction on X. Hence we have $X = x_i$. On the other hand, we have $\mathrm{FV}(X) = \{x_0, x_1, \ldots, x_i\}$. Therefore $i = 1$ and we have

$$\lambda_{x_1}^{x_0}\lambda_{x_2}^{x_0x_1}\cdots\lambda_{x_n}^{x_0x_1\cdots x_{n-1}}.(x_0\cdots x_n; xM_0\cdots M_m)^* = x_0.$$

Note that x_0x_1 is the unique term that yields x_0 as the result of abstraction $\lambda_{x_1}^{x_0}$. Thus we have

$$\lambda_{x_2}^{x_0x_1}\lambda_{x_3}^{x_0x_1x_2}\cdots\lambda_{x_n}^{x_0x_1\cdots x_{n-1}}.(x_0\cdots x_n; xM_0\cdots M_m)^* = x_0x_1.$$

Similarly we have

$$\lambda_{x_3}^{x_0x_1x_2}\cdots\lambda_{x_n}^{x_0x_1\cdots x_{n-1}}.(x_0\cdots x_n; xM_0\cdots M_m)^* = x_0x_1x_2.$$

Continuing this argument, we have

$$(x_0\cdots x_n; xM_0\cdots M_m)^* = x_0x_1x_2\cdots x_n.$$

On the other hand, we have

$$(x_0\cdots x_n; xM_1\cdots M_m)^* = x(x_0x_1\cdots x_n; M_0)^*\cdots(x_0x_1\cdots x_n; M_m)^*.$$

Thus we have $x = x_0$, $m = n-1$ and $(x_0x_1\cdots x_n; M_i)^* = x_{i+1}$ for each $0\le i\le n-1$. It follows that

$$M = \lambda x_0x_1\cdots x_n.x_0M_0\cdots M_{n-1} \xrightarrow{\eta}\!\!\!\!\rightarrow \lambda x_0x_1\cdots x_n.x_0x_1\cdots x_n \xrightarrow{\eta}\!\!\!\!\rightarrow \lambda x_0.x_0.$$

Finally we can prove that for a linear λ-term M in β-normal form contained in $\mathrm{HRM}(x_0x_1\cdots x_n)$, if $(x_0x_1\cdots x_n; M)^* = x_i$ then $M \xrightarrow{\eta}\!\!\!\!\rightarrow x_i$ by induction on M, from which the claim holds.

(3) $\Rightarrow$ (1): Let M be a β-normal form linear λ-term in $\mathrm{HRM}(x_0x_1\cdots x_n)$ such that $\mathrm{FV}(M) = \{x_0, x_1, \ldots, x_n\}$ and the η-normal form of M does not contain any subterm of the form $P(QR)$. We can prove by induction on the number of λ's of M, that $M \xrightarrow{\eta}\!\!\!\!\rightarrow x_0x_1\cdots x_n$. ⊣

§3. Proof of the main theorem. Now we prove the main theorem of this paper, without using Martin's Theorem.

THEOREM 3 (Main theorem). *If a closed hereditary right-maximal linear λ-term has type $\alpha\to\alpha$, then it is $\beta\eta$-reducible to $\lambda x.x$.*

PROOF. Let M be a closed HRML λ-term of the type $\alpha\to\alpha$. We prove the theorem by induction on the size $|\alpha|$ of the type α, which is the total number of occurrences of type variables in α. Since M is linear, M has a $\beta\eta$-normal form M'.

BASE STEP. $\alpha = a$ is a type variable. Since M' is closed, M' has the form $M' = \lambda x.N$ and the type assignment for $M' : a \to a$ has the following form.

$$\dfrac{\begin{array}{c} x : a \\ \vdots \\ N : a \end{array}}{\lambda x.N : a \to a}$$

Since a is a type variable, N is not a λ-abstraction. Since $\mathrm{FV}(N) = \{x\}$, $N = xN_1 \cdots N_m$. Since the predicate of $x : a$ is a type variable, we have $m = 0$. Hence $N = x$. Thus we have $M \xrightarrow{\beta\eta} M' = \lambda x.x$.

INDUCTION STEP. Let $M' = \lambda x_0 x_1 \cdots x_n.x_k M_1 \cdots M_m$ be the $\beta\eta$-normal form of M. Then the type assignment for $M' : \alpha \to \alpha$ has the following form.

$$\dfrac{\dfrac{x_k : \overbrace{\xi_1 \to \cdots \to \xi_m \to \gamma}^{\alpha_k} \qquad M_1 : \xi_1 \quad \cdots \quad M_m : \xi_m}{x_k M_1 \cdots M_m : \gamma}}{\lambda x_0 x_1 \cdots x_n.x_k M_1 \cdots M_m : \alpha \to \underbrace{\alpha_1 \to \cdots \to \alpha_n \to \gamma}_{\alpha}}$$

Here $\alpha_k = \xi_1 \to \cdots \to \xi_m \to \gamma$ and $\alpha = \alpha_1 \to \cdots \to \alpha_n \to \gamma$.

CASE 1. $m = 0$. Then $M' = \lambda x_0 x_1 \cdots x_n.x_k$. Since M' is closed and linear, it follows that $n = k = 0$. Hence $M' = \lambda x_0.x_0$ and Theorem holds.

CASE 2. $m \neq 0$.

SUBCASE 2.1. $n = 0$. Then $k = 0$ and $M' = \lambda x_0.x_0 M_1 \cdots M_m$. Since M' is closed and linear, $M_1, \dots, M_m$ are closed λ-terms. Therefore $\mathrm{idx}(M_1, x_0) = \cdots = \mathrm{idx}(M_m, x_0) = 0$. On the other hand, we have $1 = \mathrm{idx}(x_0, x_0) \leq \mathrm{idx}(M_1, x_0) \leq \cdots \leq \mathrm{idx}(M_m, x_0)$ by the hereditary right-maximality of M'. A contradiction. Therefore this subcase does not happen.

SUBCASE 2.2. $n \neq 0$.

SUBCASE 2.2.1. $k = 0$. Then $\alpha_k = \alpha$. Therefore

$$\xi_1 \to \cdots \to \xi_m \to \gamma = \alpha_1 \to \cdots \to \alpha_n \to \gamma.$$

Hence $m = n$ and $\xi_i = \alpha_i (i = 1, \dots, n)$. By the right-maximality of M', we have $1 = \mathrm{idx}(x_0, \vec{x}) \leq \mathrm{idx}(M_1, \vec{x}) \leq \cdots \leq \mathrm{idx}(M_m, \vec{x})$, where $\vec{x}$ is $x_0 x_1 \cdots x_n$. Since $1 \leq \mathrm{idx}(M_i, \vec{x})$, the index is attained by a variable in $x_0, x_1, \dots, x_n$ and M_i contains such a free variable. Since M' is linear, each variable occurs exactly once. Thus we have $\mathrm{idx}(M_1, \vec{x}) < \cdots < \mathrm{idx}(M_n, \vec{x})$. Therefore $\mathrm{FV}(M_i) = \{x_i\}$ $(i = 1, \dots, n)$. Thus we have $\lambda x_i.M_i : \alpha_i \to \alpha_i$. By induction hypothesis for α_i, we have $\lambda x_i.M_i \xrightarrow{\beta\eta} \lambda x_i.x_i$. Since M' is in $\beta\eta$-normal form, so is M_i. Therefore $M_i = x_i$. Thus we have $M' = \lambda x_0 x_1 \cdots x_n.x_0 x_1 \cdots x_n$. This contradicts that M' is in η-normal form. Thus this subcase does not happen.

SUBCASE 2.2.2. $k \neq 0$. Since M' is linear, x_0 occurs in some M_j.

SUBCASE 2.2.2.1. $\mathrm{FV}(M_j) = \{x_0\}$. Then $\mathrm{idx}(M_j, x_0 x_1 \cdots x_n) = 1$. This contradicts $k + 1 = \mathrm{idx}(x_k, \vec{x}) \leq \mathrm{idx}(M_1, \vec{x}) \leq \cdots \leq \mathrm{idx}(M_j, \vec{x})$. Therefore this subcase does not happen.

SUBCASE 2.2.2.2. $\mathrm{FV}(M_j) \supset \{x_0\}$. Let $\mathrm{FV}(M_j) = \{x_0, x_{j_1}, \dots, x_{j_q}\}$ with $0 < j_1 < \cdots < j_q$, $1 \le q$. Let $Y = \lambda x_0 x_{j_1} \cdots x_{j_q}.M_j$. Then Y is a closed HRML λ-term and has the following type assignment.

$$\frac{M_j : \xi_j}{Y = \lambda x_0 x_{j_1} \cdots x_{j_q}.M_j : \alpha \to \underbrace{\alpha_{j_1} \to \cdots \to \alpha_{j_q} \to \xi_j}_{\delta}}$$

Here $\delta = \alpha_{j_1} \to \cdots \to \alpha_{j_q} \to \xi_j$. Then we see that $|\delta| < |\alpha|$ as follows. Since $\alpha_k = \xi_1 \to \cdots \to \xi_j \to \cdots \to \xi_m \to \gamma$, ξ_j is a proper subtype of α_k. Therefore

$$|\delta| = |\alpha_{j_1}| + \cdots + |\alpha_{j_q}| + |\xi_j| < |\alpha_{j_1}| + \cdots + |\alpha_{j_q}| + |\alpha_k|.$$

Since M' is linear, x_k occurs exactly once. Therefore $x_{j_1}, \dots, x_{j_q}, x_k$ are distinct variables among $x_1, \dots, x_n$. Hence

$$\begin{aligned} |\alpha_{j_1}| + \cdots + |\alpha_{j_q}| + |\alpha_k| &\le |\alpha_1| + \cdots + |\alpha_n| \\ &< |\alpha_1| + \cdots + |\alpha_n| + |\gamma| \\ &< |\alpha|. \end{aligned}$$

Therefore $|\delta| < |\alpha|$.

SUBCASE 2.2.2.2.1. M_j does not contain a subterm of the form $P(QR)$. Then $Y = \lambda x_0.x_0$ by Theorem 2. This contradicts that $q \ge 1$. Thus this subcase does not happen.

SUBCASE 2.2.2.2.2. M_j contains a subterm of the form $P(QR)$. Then $Y = \lambda x_0 x_{j_1} \cdots x_{j_q}.M_j$ contains $P(QR)$, and is a closed HRML λ-term in β-normal form. Let $X = \lambda y x_1 \cdots x_n.x_k M_1 \cdots (y x_{j_1} \cdots x_{j_q}) \cdots M_m$. Then X contains a subterm of the form $P(QR)$, and is a closed HRML λ-term in β-normal form and has the following type assignment.

$$\frac{\dfrac{x_k : \xi_1 \to \cdots \to \xi_m \to \gamma \quad \cdots \quad \dfrac{y : \delta \quad x_{j_1} : \alpha_{j_1} \quad \cdots \quad x_{j_q} : \alpha_{j_q}}{y x_{j_1} \cdots x_{j_q} : \xi_j} \quad \cdots \quad M_m : \xi_m}{x_k M_1 \cdots (y x_{j_1} \cdots x_{j_q}) \cdots M_m : \gamma}}{X = \lambda y x_0 \cdots x_n.x_k M_1 \cdots (y x_{j_1} \cdots x_{j_q}) \cdots M_m : \delta \to \underbrace{\alpha_1 \to \cdots \to \alpha_n \to \gamma}_{\alpha}}$$

By Theorem 2, X^+ and Y^+ consist only of B and B'. Let Z be $\lambda x.Y(Xx)$. This is a closed HRML λ-term of type $\delta \to \delta$. Moreover, Z^+ consists only of B and B'. Since $Z^{+\diamond}$ consists only of $B^\diamond$ and $B'^\diamond$, it is in η-normal form and contains a subterm of the form $P(QR)$. Now we use two important properties of β-reductions with respect to $B^\diamond = \lambda xyz.x(yz)$ and $B'^\diamond = \lambda xyz.y(xz)$: namely, they preserve any subterm of form $P(QR)$, and never create a new η-redex. Therefore $Z^{+\diamond} \not\twoheadrightarrow_{\beta\eta} \lambda x.x$. Hence by Theorem 1, $Z \not\twoheadrightarrow_{\beta\eta} \lambda x.x$, which, however, contradicts to the induction hypothesis.[2] Thus this subcase does not happen. ⊣

[2]We can obtain a contradiction also by the characterization theorem for $\lambda\eta$-invertibility (Barendregt [2], Dezani [3]) as follows: Since X is linear and contains $P(QR)$, X is not a finite hereditary permutation. Thus X is not $\lambda\eta$-invertible. Hence in particular, $Z = \lambda x.Y(Xx) \not\twoheadrightarrow_{\beta\eta} \lambda x.x$. But Z is of type $\delta \to \delta$, which contradicts to the induction hypothesis.

§4. Proof of Martin's Theorem by λ-calculus. Now we prove Martin's theorem as a corollary to Theorem 3.

THEOREM 4 (Martin and Meyer [8]). *If both $\alpha \to \beta$ and $\beta \to \alpha$ are provable in P-W, then $\alpha = \beta$.*

PROOF. Under the assumption that formulas $\alpha \to \beta$ and $\beta \to \alpha$ are provable in P-W, we choose closed HRML λ-terms M and N in β-normal form of type $\alpha \to \beta$ and $\beta \to \alpha$, respectively. If neither M^+ nor N^+ contains I, by defining Z as $\lambda x.N(Mx)$ of type $\beta \to \beta$, we obtain a contradiction to Theorem 3 by exactly following the argument in Subcase 2.2.2.2.2. Therefore either M^+ or N^+ contains an I, hence by Theorem 2, $\alpha = \beta$ holds. ⊣

Now we directly derive, from Theorem 3, the statement due to Powers, known as being equivalent to Martin's theorem.

THEOREM 5. *No formula of the form $\alpha \to \alpha$ is provable in P-W without I.*

PROOF. Under the assumption that formula $\alpha \to \alpha$ is provable in P-W, there exists a BB'-combinator X of type $\alpha \to \alpha$. Therefore a closed HRML λ-term $X^\diamond$ has type $\alpha \to \alpha$, and, by Theorem 3, $X^\diamond \xrightarrow{\beta\eta} \lambda x.x$. However, since $X^\diamond$ is made of $B^\diamond = \lambda xyz.x(yz)$ and $B'^\diamond = \lambda xyz.y(xz)$, a subterm of the form $P(QR)$ remains after any reduction. A contradiction. Therefore $\alpha \to \alpha$ is not provable in P-W without I. ⊣

Finally, we directly derive, from Theorem 3, the statement due to Powers and Dwyer, known as being equivalent to Martin's theorem.

THEOREM 6. *$\alpha \to \beta$ is provable in P-W without I iff $\alpha \to \beta$ is provable in P-W and $\alpha \neq \beta$.*

PROOF. The only-if-part is immediate from Theorem 5. So, it suffices to prove the if-part. Under the assumption that formula $\alpha \to \beta$ is provable in P-W and $\alpha \neq \beta$ we choose a closed HRML λ-term M in β-normal form such that $M : \alpha \to \beta$. If the translated $BB'I$-combinator M^+ contains an I, then $M \xrightarrow{\eta} \lambda x.x$ by Theorem 2. This contradicts $\alpha \neq \beta$. Therefore M^+ does not contain I. Thus M^+ is made of B and B'. Therefore $\alpha \to \beta$ is provable in P-W without I. ⊣

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