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LOGICS WITHOUT THE CONTRACTION RULE

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§1. Introduction. We will study syntactical and semantical properties of propositional logics weaker than the intuitionistic, in which the contraction rule (or, the exchange rule or the weakening rule, in some cases) does not hold. Here, the contraction rule means the rule of inference of the form

$$\frac{\Gamma, \alpha, \alpha, \Delta \to \gamma}{\Gamma, \alpha, \Delta \to \gamma}$$

if we formulate our logics in a Gentzen-type formal system. Some syntactical properties of these logics have been studied firstly by the second author in [11], in connection with the study of BCK-algebras (for information on BCK-algebras, see [9]). There, it turned out that such a syntactical method is a powerful and promising tool in studying BCK-algebras. Using this method, considerable progress has been made since then (see, e.g., [8], [18], [27]).

In this paper, we will study these logics more comprehensively. We notice here that the distributive law

$$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

does not hold necessarily in these logics. By adding some axioms (or initial sequents) and rules of inference to these basic logics, we can obtain a lot of interesting nonclassical logics such as Łukasiewicz's many-valued logics, relevant logics, the intuitionistic logic and logics related to BCK-algebras, which have been studied separately until now. Thus, our approach will give a uniform way of dealing with these logics. One of our two main tools in doing so is Gentzen-type formulation of logics in syntax, and the other is semantics defined by using partially ordered monoids.

In §2, we will introduce Gentzen-type systems for three basic logics, for each of which the cut elimination theorem holds. This fact will clarify what is meant by logics without the contraction rule (or the exchange rule), and therefore will explain the connection of them with their semantics developed in §3 in a more explicit way, though this connection has been pointed out already by Urquhart [23].

We will define semantics for these logics by using partially ordered monoids in §3, and will prove the completeness theorem in §4. We have gotten a hint of our

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semantics from Idziak's work [8]. Our semantics has some resemblances to semantics for relevant logics by Routley and Meyer [20], Urquhart [23] and, in particular, Fine [2] (see §7).[1] But it should be noticed that unlike relevant logics, the distributive law does not hold always in our logics and hence the argument using *prime theories*, which plays an important role in proving the completeness theorem in [2] or [20], does not work well in our case.

It will be shown that our semantics can cover a wide class of logics, for example, logics satisfying the distributive law (§5) and many-valued logics, logics without the weakening rule and relevant logics (§7). Moreover, in case of the intuitionistic logic, it will be shown in §6 that our semantics coincides with Kripke's original semantics in [14]. As an application of these logical results, we will show in §8 some embedding theorems for BCK- and related algebras.

§2. Formal systems for logics without the contraction rule. In our formalization, we will take $\supset$ (implication), $\vee$ (disjunction) and two kinds of conjunction $\wedge$ and & as logical connectives, and $\bot$ (falsehood) as a propositional constant. As shown below, both $\vee$ and $\wedge$ are contractible and exchangeable, while & is neither. We remark here that the usual distributive law between $\vee$ and $\wedge$, i.e.,

$$\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

does not necessarily hold (see §5). On the other hand, the following distributive law between $\vee$ and &

$$\alpha \,\&\, (\beta \vee \gamma) \to (\alpha \,\&\, \beta) \vee (\alpha \,\&\, \gamma)$$

holds, while

$$\alpha \vee (\beta \,\&\, \gamma) \to (\alpha \vee \beta) \,\&\, (\alpha \vee \gamma)$$

does not.

Formulas are defined in the usual way. We use Latin letters $p, q, r, \ldots$ for propositional variables and Greek letters $\alpha, \beta, \gamma, \ldots$ for formulas. We abbreviate $\alpha \supset \bot$ as $\neg\alpha$. Sometimes we will omit parentheses, subject to the convention that $\supset$ has the weakest order of priority. For example, $\alpha \,\&\, \beta \supset \gamma$ is to be read as $(\alpha \,\&\, \beta) \supset \gamma$. We also adopt the convention of association to the right for omitting parentheses. Thus, $\alpha \wedge \beta \wedge \gamma$ is to be read as $\alpha \wedge (\beta \wedge \gamma)$, $\alpha \supset \beta \supset \gamma$ is to be read as $\alpha \supset (\beta \supset \gamma)$, and so on.

Now we will introduce a basic logical system called L_{BCC}, which is formulated in a Gentzen-type system. Roughly speaking, L_{BCC} is a formal system obtained from Gentzen's system *LJ* for the intuitionistic logic, by eliminating both the contraction rule and the exchange rule. In the following, we will use Greek capital letters $\Gamma, \Delta, \Sigma, \ldots$ for finite (possibly empty) sequences of formulas separated by commas. An expression of the form $\Gamma \to \delta$ is called a sequent, where Γ is a finite sequence of formulas and δ is a formula. We assume here a familiarity with the basic knowledge of Gentzen-type systems (see e.g. [22]).

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[1] The authors wish to thank a referee, who after reading the first draft of this paper pointed out the connection between our semantics and semantics for relevant and many-valued logics, studied by Fine [2], Routley and Meyer [20] and Urquhart [23], [24].

Initial sequents of L_{BCC} are either of the form $\perp \to \alpha$ for any formula α, or of the form $p \to p$ for any propositional variable p. Rules of inferences of L_{BCC} are as follows;

$$\frac{\Gamma, \Delta \to \gamma}{\Gamma, \alpha, \Delta \to \gamma}\text{(weakening)} \qquad \frac{\Gamma \to \alpha \quad \Delta, \alpha, \Sigma \to \gamma}{\Delta, \Gamma, \Sigma \to \gamma}\text{(cut)}$$

$$\frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}(\to \supset) \qquad \frac{\Gamma \to \alpha \quad \Delta, \beta, \Sigma \to \gamma}{\Delta, \alpha \supset \beta, \Gamma, \Sigma \to \gamma}(\supset \to)$$

$$\frac{\Gamma \to \alpha}{\Gamma \to \alpha \vee \beta}(\to \vee 1) \qquad \frac{\Gamma \to \beta}{\Gamma \to \alpha \vee \beta}(\to \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \to \gamma \quad \Gamma, \beta, \Delta \to \gamma}{\Gamma, \alpha \vee \beta, \Delta \to \gamma}(\vee \to)$$

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}(\to \wedge)$$

$$\frac{\Gamma, \alpha, \Delta \to \gamma}{\Gamma, \alpha \wedge \beta, \Delta \to \gamma}(\wedge \to 1) \qquad \frac{\Gamma, \beta, \Delta \to \gamma}{\Gamma, \alpha \wedge \beta, \Delta \to \gamma}(\wedge \to 2)$$

$$\frac{\Gamma \to \alpha \quad \Delta \to \beta}{\Gamma, \Delta \to \alpha \& \beta}(\to \&) \qquad \frac{\Gamma, \alpha, \beta, \Delta \to \gamma}{\Gamma, \alpha \& \beta, \Delta \to \gamma}(\& \to).$$

Next, we will take the following two rules;

$$\frac{\Gamma, \alpha, \beta, \Delta \to \gamma}{\Gamma, \beta, \alpha, \Delta \to \gamma}\text{(exchange)} \qquad \frac{\Gamma, \alpha, \alpha, \Delta \to \gamma}{\Gamma, \alpha, \Delta \to \gamma}\text{(contraction).}$$

The formal system L_{BCK} (or LJ^*) is obtained from L_{BCC} by adding the exchange rule (or, the exchange rule and the contraction rule, respectively). The relation between L_{BCK} (or L_{BCC}) and BCK- (or BCC-) algebras will be clarified in §8. We notice that the position of formulas in sequents in rules of L_{BCC}, in particular, the position of Γ and Δ in lower sequents of $(\supset \to)$ and $(\to \&)$, is significant, since the exchange rule is not admitted in L_{BCC}.

We can show that 1) neither $\alpha \& \beta \to \beta \& \alpha$ nor $\alpha \to \alpha \& \alpha$ is provable in L_{BCC}, while $\alpha \wedge \beta \to \beta \wedge \alpha$ and $\alpha \to \alpha \wedge \alpha$ are both provable in it, and that 2) in L_{BCC} $\alpha \& \beta \to \alpha \wedge \beta$ is provable but $\alpha \wedge \beta \to \alpha \& \beta$ is not, while the latter is provable in LJ^*. Moreover, we can verify the following lemmas without difficulty.

LEMMA 2.1. *The following three conditions are equivalent.*

(1) $\alpha_1, \alpha_2, \ldots, \alpha_n \to \beta$ *is provable in* L_{BCC}.

(2) $\to \alpha_1 \supset \alpha_2 \supset \cdots \supset \alpha_n \supset \beta$ *is provable in* L_{BCC}.

(3) $\to \alpha_1 \& \alpha_2 \& \cdots \& \alpha_n \supset \beta$ *is provable in* L_{BCC}.

LEMMA 2.2. 1) *Let* δ (*or* Γ) *be a formula* (*or a sequence of formulas*) *containing no* &. Then, the sequent $\Gamma \to \delta$ *is provable in* LJ^* *if and only if it is provable in* LJ.

2) *Let* L^+_{BCC} *be the formal system obtained from* L_{BCC} *by adding only the contraction rule to it. Then, the exchange rule is derivable in* L^+_{BCC}, *but the cut elimination theorem does not hold for* L^+_{BCC} (*cf.* Theorem 2.3).

THEOREM 2.3. *The cut elimination theorem holds for* L_{BCC}, L_{BCK} *and* LJ^*.

PROOF. Our theorem can be proved in the standard way (see [22]). So, we will give here a sketch of the proof of the cut elimination theorem only for L_{BCC}. Modifying things slightly, we can show also the cut elimination theorem for the other two logics. In proving our theorem, we must keep in mind that L_{BCC} (or L_{BCK}) does not have the contraction rule. But this fact will simplify the proof. For, we need not replace each cut rule by a *mix rule*. Thus, it suffices to show that each *cut rule* in a given proof is eliminable. As Lemma 5.4 of [22], we have only to show that

(*) *If P is a proof of a sequent S, which contains only one cut rule, occurring as the last inference, then S is provable without a cut.*

So, let us suppose that P is a proof which contains a cut only as the last inference:

$$\frac{\Gamma \to \alpha \qquad \Delta, \alpha, \Sigma \to \gamma}{\Delta, \Gamma, \Sigma \to \gamma}.$$

The *grade* of the proof P is the number of occurrences of logical connectives in the cut formula α. The *rank* of the proof P is the number of sequents appearing in P. As usual, (*) can be proved by double induction on the grade and the rank of P. We divide the proof into the following four cases:

Case 1. Either $\Gamma \to \alpha$ or $\Delta, \alpha, \Sigma \to \gamma$ is an initial sequent.

Case 2. Either $\Gamma \to \alpha$ or $\Delta, \alpha, \Sigma \to \gamma$ is a lower sequent of a weakening rule.

Case 3. Either $\Gamma \to \alpha$ or $\Delta, \alpha, \Sigma \to \gamma$ is a lower sequent of a logical rule, whose principal formula is not (the occurrence of) α, to which the cut rule is applied.

Case 4. Both $\Gamma \to \alpha$ and $\Delta, \alpha, \Sigma \to \gamma$ are lower sequents of some logical rules such that principal formulas of both rules are (occurrences of) α to which the cut rule is applied.

In the following, we will show (*) only for Case 4 where α is of the form $\beta \supset \delta$. In this case, the last part of P is of the form

$$\frac{\dfrac{\Gamma, \beta \to \delta}{\Gamma \to \beta \supset \delta} \qquad \dfrac{\Sigma \to \beta \qquad \Delta, \delta, \Pi \to \gamma}{\Delta, \beta \supset \delta, \Sigma, \Pi \to \gamma}}{\Delta, \Gamma, \Sigma, \Pi \to \gamma}\text{(cut)}.$$

Consider the following proof P':

$$\frac{\begin{matrix}\vdots \\ \Sigma \to \beta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \Gamma, \beta \to \delta\end{matrix} \qquad \begin{matrix}\vdots \\ \Delta, \delta, \Pi \to \gamma\end{matrix}}{\Delta, \Gamma, \beta, \Pi \to \gamma}\text{(cut)}}{\Delta, \Gamma, \Sigma, \Pi \to \gamma}\text{(cut)}.$$

Clearly, the end-sequent of P' is the same as that of P. Moreover, both cut formulas in P' contain fewer logical symbols than $\beta \supset \delta$. Thus, by the induction hypothesis, we can eliminate these cut rules.

Similarly to the proof of Theorem 6.6 of [22], we can derive the following theorem from Theorem 2.3.

THEOREM 2.4. *Craig's interpolation theorem holds for* L_{BCC}, L_{BCK} *and* LJ^*. *More precisely, if a sequent* $\Delta, \Gamma, \Sigma \to \gamma$ *is provable in L then there exists a formula* α *such that*

(1) $\Gamma \to \alpha$ *and* $\Delta, \alpha, \Sigma \to \gamma$ *are both provable in L, and*

(2) *all propositional variables in* α *occur both in* Γ *and* $\Delta \cup \Sigma \cup \{\gamma\}$, *where L is any of* L_{BCC}, L_{BCK} *and* LJ^*.

Let C be an arbitrary subset of the set of logical symbols $\{\perp, \supset, \vee, \wedge, \&\}$ containing at least $\supset$. A formula α is a *C-formula*, if any logical symbol appearing in α belongs to C. A sequent consisting only of C-formulas is called a *C-sequent*. Let L be any of L_{BCC}, L_{BCK} and LJ^*. Then the *C-fragment* of L is the formal system dealing with only C-sequents, whose initial sequents are $p \to p$ for any propositional variable p, and $\perp \to \alpha$ for any C-formula α only when $\perp \in C$, and whose rules of inference are structural rules of L and rules for logical symbols in C.

By observing the proof of Theorem 2.4, we have the following.

COROLLARY 2.5. *Let* C *be either* $\{\perp, \supset, \&\}$ *or* $\{\perp, \supset, \wedge, \&\}$. *Then Craig's interpolation theorem of the form mentioned in Theorem 2.4 holds for C-fragments of* L_{BCC}, L_{BCK} *and* LJ^*.

Theorem 2.4 and Corollary 2.5 for L_{BCK} are proved independently by P. Idziak. On the other hand, Craig's interpolation theorem does not hold for the implicational-fragment, i.e. $\{\supset\}$-fragment, of these logics, in the above form. But in [27] Wronski proved Craig's interpolation theorem for the (dual of the) implicational-fragment of L_{BCK} in another form.

We will next introduce Hilbert-style formal systems H_{BCC} and H_{BCK}, which are logically equivalent to L_{BCC} and L_{BCK}, respectively. Moreover, we will show that the separation theorem holds for H_{BCK}. Here, we say that the separation theorem holds for a Hilbert-style formal system L, if for any provable formula in L there exists a proof of it which uses only the axioms for implication and the axioms for the other logical symbols actually appearing in the formula. It is well known that both the classical and the intuitionistic logics can be formalized in Hilbert-style formal systems such that the separation theorem holds in them (see [4] and [5]).

The formal system H_{BCC} has two kinds of modus ponens

$$\frac{\alpha \quad \alpha \supset \beta}{\beta}\text{(m.p. 1)}, \qquad \frac{\alpha \quad \beta \supset \alpha \supset \gamma}{\beta \supset \gamma}\text{(m.p. 2)},$$

and the following twelve axiom schemata:

$\alpha \supset \beta \supset \alpha$	(H weakening),
$\perp \supset \alpha$	($H \perp$),
$(\alpha \supset \beta) \supset (\gamma \supset \alpha) \supset \gamma \supset \beta$	($H \supset$),
$(\alpha \supset \gamma) \wedge (\beta \supset \gamma) \supset \alpha \vee \beta \supset \gamma$	($H \vee l$),
$\alpha \supset \alpha \vee \beta$	($H \vee r1$),
$\beta \supset \alpha \vee \beta$	($H \vee r2$),
$\alpha \wedge \beta \supset \alpha$	($H \wedge l1$),
$\alpha \wedge \beta \supset \beta$	($H \wedge l2$),
$(\gamma \supset \alpha) \wedge (\gamma \supset \beta) \supset \gamma \supset \alpha \wedge \beta$	($H \wedge r1$),
$\alpha \supset \beta \supset \alpha \wedge \beta$	($H \wedge r2$),
$(\alpha \supset \beta \supset \gamma) \supset \alpha \& \beta \supset \gamma$	($H \& l$),
$\alpha \supset \beta \supset \alpha \& \beta$	($H \& r$).

The formal system H_{BCK} is obtained from H_{BCC} by first eliminating $(H \vee l)$ and then adding the following two axiom schemata:

$$(\alpha \supset \gamma) \supset (\beta \supset \gamma) \supset \alpha \vee \beta \supset \gamma \qquad (H \vee l'),$$
$$(\alpha \supset \beta \supset \gamma) \supset \beta \supset \alpha \supset \gamma \qquad (H \text{ exchange}).$$

It is easy to see that (m.p. 2) is superfluous in H_{BCK} since it can derived from (m.p. 1) by using (H exchange).

Next we will introduce auxiliary Gentzen-type formal systems L^*_{BCC} and L^*_{BCK}. Corresponding to axiom schemata of H_{BCC}, initial sequents of L^*_{BCC} are given as follows:

$$\alpha, \beta \to \alpha \qquad (\text{weakening*}),$$
$$\bot \to \alpha \qquad (\bot),$$
$$(\alpha \supset \gamma) \wedge (\beta \supset \gamma), \alpha \vee \beta \to \gamma \qquad (\vee\, l),$$
$$\alpha \to \alpha \vee \beta \qquad (\vee\, r1),$$
$$\beta \to \alpha \vee \beta \qquad (\vee\, r2),$$
$$\alpha \wedge \beta \to \alpha \qquad (\wedge\, l1),$$
$$\alpha \wedge \beta \to \beta \qquad (\wedge\, l2),$$
$$(\gamma \supset \alpha) \wedge (\gamma \supset \beta), \gamma \to \alpha \wedge \beta \qquad (\wedge\, r1),$$
$$\alpha, \beta \to \alpha \wedge \beta \qquad (\wedge\, r2),$$
$$\alpha \supset \beta \supset \gamma, \alpha \,\&\, \beta \to \gamma \qquad (\&\, l),$$
$$\alpha, \beta \to \alpha \,\&\, \beta \qquad (\&\, r).$$

Rules of inference of L^*_{BCC} are the following:

$$\frac{\to \alpha \qquad \alpha, \Sigma \to \gamma}{\Sigma \to \gamma}(\text{cut } 1), \qquad \frac{\to \alpha \qquad \delta, \alpha, \Sigma \to \gamma}{\delta, \Sigma \to \gamma}(\text{cut } 2),$$

$$\frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}(\to \supset),$$

$$\frac{\beta, \Sigma \to \gamma}{\alpha \supset \beta, \alpha, \Sigma \to \gamma}(\supset \to 1), \qquad \frac{\delta, \beta, \Sigma \to \gamma}{\delta, \alpha \supset \beta, \alpha, \Sigma \to \gamma}(\supset \to 2).$$

Initial sequents of L^*_{BCK} are those of L^*_{BCC} except $(\vee\, l)$, and the following:

$$\alpha \supset \beta \supset \gamma, \beta, \alpha \to \gamma \qquad (\text{exchange*}),$$
$$\alpha \supset \gamma, \beta \supset \gamma, \alpha \vee \beta \to \gamma \qquad (\vee\, l').$$

Rules of inferences of L^*_{BCK} are the same as those of L^*_{BCC}.

Let C be any subset of the set of logical symbols $\{\bot, \supset, \vee, \wedge, \&\}$ containing at least $\supset$. The C-fragment of H_{BCC} (or H_{BCK}) is a Hilbert-style formal system dealing with only C-formulas whose rules are two kinds of modus ponens and whose axioms are (H weakening) (or, (H weakening) and (H exchange)) and axiom schemata of H_{BCC} (or H_{BCK}, respectively) for logical symbols in C. Similarly, the C-fragment of L^*_{BCC} (or L^*_{BCK}) is a Gentzen-type formal system dealing with only C-sequents, whose rules are those of L^*_{BCC} and whose initial sequents are (weakening*) (or, (weakening*) and (exchange*)) and those of L^*_{BCC} (or L^*_{BCK}, respectively), concerning logical symbols in C.

THEOREM 2.6. *L^*_{BCC} and H_{BCC} are logically equivalent, that is, for each formula $\alpha_1, \ldots, \alpha_n, \delta$, the sequent $\alpha_1, \ldots, \alpha_n \to \delta$ is provable in L^*_{BCC} if and only if the formula $\alpha_1 \supset \cdots \supset \alpha_n \supset \delta$ is provable in H_{BCC}. More exactly, for any subset C of logical symbols containing at least $\supset$, the C-fragment of L^*_{BCC} and the C-fragment of H_{BCC} are logically equivalent, that is, for each C-formula $\alpha_1, \ldots, \alpha_n, \delta$, $\alpha_1, \ldots, \alpha_n \to \delta$ is provable in the C-fragment of L^*_{BCC} if and only if $\alpha_1 \supset \cdots \supset \alpha_n \supset \delta$ is provable in the C-fragment of H_{BCC}. The similar result holds also for L^*_{BCK} and H_{BCK}.*

THEOREM 2.7. *L_{BCK} and L^*_{BCK} are logically equivalent, that is, for any sequent $\Gamma \to \delta$, $\Gamma \to \delta$ is provable in L_{BCK} if and only if it is provable in L^*_{BCK}. More exactly, for any subset C of logical symbols containing at least $\supset$, the C-fragment of L_{BCK} and the C-fragment of L^*_{BCK} are logically equivalent, that is, for any C-sequent $\Gamma \to \delta$, $\Gamma \to \delta$ is provable in the C-fragment of L_{BCK} if and only if it is provable in the C-fragment of L^*_{BCK}. The similar result holds also for L_{BCC} and L^*_{BCC} under the condition that either C does not contain $\vee$ or C contains $\wedge$.*

PROOF. Our theorem can be obtained by proving that every initial sequent of the C-fragment of L^*_{BCK} (or L^*_{BCC}) is provable in the C-fragment of L_{BCK} (or L_{BCC}) and every rule of inference of the C-fragment of L^*_{BCK} (or L^*_{BCC}) is a derived rule of the C-fragment of L_{BCK} (or L_{BCC} for C satisfying the condition mentioned in our theorem, respectively) and vice versa. Here we will prove only that the cut rule in L_{BCC} is a derived rule in L^*_{BCC}. First, we will show that the following restricted form of the cut rule

$$\text{(1)} \qquad \frac{\Gamma \to \alpha \qquad \alpha, \Sigma \to \gamma}{\Gamma, \Sigma \to \gamma}$$

is derivable in L^*_{BCC}. Let Γ be $\beta_1, \ldots, \beta_n$. We will show (1) by induction on n. When $n = 0$, (1) is nothing but (cut 1) in L^*_{BCC}. Suppose that $n > 0$. Then,

$$\cfrac{\cfrac{\beta_1, \ldots, \beta_n \to \alpha}{\beta_1, \ldots, \beta_{n-1} \to \beta_n \supset \alpha}(\to \supset) \qquad \cfrac{\alpha, \Sigma \to \gamma}{\beta_n \supset \alpha, \beta_n, \Sigma \to \gamma}(\supset \to 1)}{\beta_1, \ldots, \beta_n, \Sigma \to \gamma}\text{(ind. hyp.)}.$$

Similarly,

$$\text{(2)} \qquad \frac{\Gamma \to \alpha \qquad \delta, \alpha, \Sigma \to \gamma}{\delta, \Gamma, \Sigma \to \gamma}$$

is shown to be derivable in L^*_{BCC}, by using (cut 2) and ($\supset \to 2$), in place of (cut 1) and ($\supset \to 1$), respectively. Now, we will show that the cut rule

$$\text{(3)} \qquad \frac{\Gamma \to \alpha \qquad \Delta, \alpha, \Sigma \to \gamma}{\Delta, \Gamma, \Sigma \to \gamma}$$

is a derived rule in L^*_{BCC}. Let Σ be $\sigma_1, \ldots, \sigma_m$. Then we abbreviate $\sigma_1 \supset \cdots \supset \sigma_m \supset \gamma$ to $\Sigma \supset \gamma$. By using ($\to \supset$),

$$\frac{\Delta, \alpha, \Sigma \to \gamma}{\Delta \to \alpha \supset \Sigma \supset \gamma}.$$

On the other hand,

$$\frac{\Gamma \to \alpha \qquad \dfrac{\gamma \to \gamma}{\alpha \supset \Sigma \supset \gamma, \alpha, \Sigma \to \gamma}\,(\supset \to 1)}{\alpha \supset \Sigma \supset \gamma, \Gamma, \Sigma \to \gamma}\,(\text{by } (2)).$$

Now by (1),

$$\frac{\Delta \to \alpha \supset \Sigma \supset \gamma \qquad \alpha \supset \Sigma \supset \gamma, \Gamma, \Sigma \to \gamma}{\Delta, \Gamma, \Sigma \to \gamma}.$$

Thus, (3) is derivable in (the implicational fragment of) L^*_{BCC}.

From Theorems 2.6 and 2.7, the next corollary follows immediately.

COROLLARY 2.8.1) L_{BCK} *and* H_{BCK} *are logically equivalent. More exactly, for any subset C of logical symbols containing at least* $\supset$, *the C-fragment of* L_{BCK} *and the C-fragment of* H_{BCK} *are logically equivalent.*

2) L_{BCC} *and* H_{BCC} *are logically equivalent. More exactly, for any subset C of logical symbols containing* $\supset$, *which does not contain* $\vee$ *or contains* $\wedge$, *the C-fragment of* L_{BCC} *and the C-fragment of* H_{BCC} *are logically equivalent.*

THEOREM 2.9. *The separation theorem holds for* H_{BCK}.

PROOF. Suppose that a formula α is provable in H_{BCK}. Let C be the set of all logical symbols appearing in α, supplemented by the implication $\supset$. Then by Corollary 2.8.1), the sequent $\to \alpha$ is provable in L_{BCK}. Moreover, by Theorem 2.3, there is a proof of $\to \alpha$ in which no cut rules are used. Then, this proof can be regarded as a proof of $\to \alpha$ in the C-fragment of L_{BCK}. Using Corollary 2.8 again, the formula α is provable in the C-fragment of H_{BCK}. Thus, our theorem follows.

§3. Semantics. In this section, we will introduce Kripke-type semantics for logics without the contraction rule.

A structure $\langle M, \cdot, 1; \leqq \rangle$ is a *PO-monoid*, if it is a partially ordered monoid having an identity which is also the least element in M; that is,

(i) $\langle M, \cdot, 1 \rangle$ is a monoid with the identity 1,

(ii) $\langle M, \leqq \rangle$ is a partially ordered set satisfying $a \leqq b$ implies $a \cdot c \leqq b \cdot c$ and $c \cdot a \leqq c \cdot b$ for all $a, b, c \in M$, and

(iii) $1 \leqq a$ for all $a \in M$.

A PO-monoid $\langle M, \cdot, 1; \leqq \rangle$ is an *SO-monoid*, if $\langle M, \leqq \rangle$ is a meet-semilattice, i.e. a partially ordered set in which any two elements have a meet denoted by $\cap$, satisfying $a \cdot (b \cap c) = a \cdot b \cap a \cdot c$ and $(b \cap c) \cdot a = b \cdot a \cap c \cdot a$ for all $a, b, c \in M$.

Hereafter, the multiplication $a \cdot b$ in a semigroup is sometimes denoted by juxtaposition ab without a dot. We remark that $a \leqq ab$ and $a \leqq ba$ for all $a, b \in M$. We say b is an *extension* of a if $a \leqq b$.

Let $\boldsymbol{M}$ be an SO-monoid $\langle M, \cdot, 1; \leqq \rangle$ and K be a nonempty subset of M satisfying the following conditions: for any $a, b, c, d \in M$,

(i) if $a \in K$ and $bdc \leqq a$ then d has an extension d' in K such that $bd'c \leqq a$, and

(ii) if $a \in K$ and $b \cap c \leqq a$ and moreover if b has some extensions in K then b has an extension $b' \in K$ such that $b' \cap c \leqq a$. If b has no extensions in K then $c \leqq a$.

Then the pair $\langle \boldsymbol{M}, K \rangle$ is called a *frame*. It is clear that M itself satisfies both (i) and (ii) for K. Thus, $\langle \boldsymbol{M}, M \rangle$ is an example of a frame. Such a frame is called a *total* frame. As another example, if $\boldsymbol{M}$ is a linearly ordered SO-monoid and K is an interval of the form either $[1, a]$ or $[1, a)$ for $a > 1$, then $\langle M, K \rangle$ is a frame. In the following, we sometimes identify an SO-monoid M with its base set M if no confusions will occur.

A *weak valuation* $\models$ on a frame $\langle M, K \rangle$ is a relation between elements of K and propositional variables such that

(a) for any $a, b, c \in K$ and any propositional variable p, if $a \models p$, $b \models p$ and $a \cap b \leqq c$, then $c \models p$.

From (a) it follows that for any $a, b \in K$

$$\text{if } a \models p \text{ and } a \leqq b \text{ then } b \models p.$$

Every weak valuation $\models$ on $\langle M, K \rangle$ can be extended to a relation between elements of K and formulas, inductively as follows: For any $a \in K$,

(b) $a \not\models \bot$,

(c) $a \models \alpha \supset \beta$ if and only if, for any $b, c \in K$, $b \models \alpha$ and $ab \leqq c$ imply $c \models \beta$,

(d) $a \models \alpha \vee \beta$ if and only if, for some $b, c \in K$ such that $a \geqq b \cap c$, both ($b \models \alpha$ or $b \models \beta$) and ($c \models \alpha$ or $c \models \beta$),

(e) $a \models \alpha \wedge \beta$ if and only if $a \models \alpha$ and $a \models \beta$, and

(f) $a \models \alpha \& \beta$ if and only if, for some $b, c \in K$ such that $a \geqq bc$, $b \models \alpha$ and $c \models \beta$.

Notice that if either $a \models \alpha$ or $a \models \beta$ then $a \models \alpha \vee \beta$. On the other hand, the converse does not hold always. A triple $\langle M, K, \models \rangle$ of a frame $\langle M, K \rangle$ and a weak valuation $\models$ on it is called a *weak model*. A formula φ is said to be *valid* in a weak model $\langle M, K, \models \rangle$ if a $\models \varphi$ for all $a \in K$.

Let $\langle M, K \rangle$ be a frame such that M has the greatest element ∞ and $\infty \in K$. Then, $\langle M, K \rangle$ is called a *strong frame*. We remark that each element of M has an extension ∞ in K, in every strong frame. A weak valuation $\models$ on a strong frame satisfying also

(a′) $\infty \models p$ for any propositional variable p,

is called a *strong valuation* if $\models$ is extended to a relation between elements of K and formulas, by using (c), (d), (e), (f) and

(b′) $a \models \bot$ if and only if $a = \infty$,

instead of (b). A triple $\langle M, K, \models \rangle$ of a strong frame $\langle M, K \rangle$ and a strong valuation $\models$ on it is called a *strong model*. The validity of formulas in a given strong model can be defined in the same way as that in a weak model. We have the following lemma, by induction on the length of φ.

LEMMA 3.1. *In every strong model, $\infty \models \varphi$ holds for each formula φ.*

LEMMA 3.2. *Let $\langle M, K, \models \rangle$ be a strong model.*

1) *For any formula φ and any $a, b, c \in K$,*

$$\text{if } a \models \varphi, b \models \varphi \text{ and } a \cap b \leqq c, \text{ then } c \models \varphi.$$

2) *For any formula φ and any $a, b \in K$,*

$$\text{if } a \models \varphi \text{ and } a \leqq b \text{ then } b \models \varphi.$$

3) *Moreover, if $\langle M, K \rangle$ is total then*

$$a \models \varphi \text{ and } b \models \varphi \text{ if and only if } a \cap b \models \varphi$$

for any formula φ and any $a, b \in K$.

PROOF. We first remark that 2) follows from 1) and 3) follows from 1) and 2). We will show 1) by induction on the length of φ. Our lemma can be easily verified when φ is either a propositional variable or $\perp$. Suppose that φ is of the form $\alpha \supset \beta$. We assume that $d \models \alpha$ and $cd \leqq e$ for $d, e \in K$. Then, $ad \cap bd = (a \cap b)d \leqq cd \leqq e$. Since ad and bd have extensions in K, there exist $f, g \in K$ such that $ad \leqq f$, $bd \leqq g$ and $f \cap g \leqq e$, by the condition (ii) for K. By using the assumption that $a \models \alpha \supset \beta$ and $b \models \alpha \supset \beta$, we have $f \models \beta$ and $g \models \beta$. Thus $e \models \beta$ by the hypothesis of induction. Hence $c \models \varphi$. Next, suppose that φ is of the form $\alpha \vee \beta$. In the following, we abbreviate the condition $x \models \alpha$ or $x \models \beta$ to $D(x)$. By the assumption, there are $d_1, d_2, e_1, e_2 \in K$ such that $a \geqq d_1 \cap d_2$, $b \geqq e_1 \cap e_2$, $D(d_1)$, $D(d_2)$, $D(e_1)$ and $D(e_2)$. Now suppose that $x \models \alpha$ for $x = d_1, d_2, e_1, e_2$. Then, by the induction hypothesis, $a \models \alpha$ and $b \models \alpha$. Thus, $D(a)$, $D(b)$ and $a \cap b \leqq c$ hold. Hence, $c \models \alpha \vee \beta$. The case where $x \models \beta$ for $x = d_1, d_2, e_1, e_2$ can be treated similarly. Suppose next that $d_1 \models \alpha$ and $x \models \beta$ for $x = d_2, e_1, e_2$. Then there is an element f in K such that $d_2 \cap e_1 \cap e_2 \leqq f$ and $d_1 \cap f \leqq c$. Similarly, there is an element e in K such that $e_1 \cap e_2 \leqq e$ and $d_2 \cap e \leqq f$. By the induction hypothesis, $e \models \beta$ and hence $f \models \beta$. Thus, $D(d_1)$, $D(f)$ and $d_1 \cap f \leqq c$. Hence $c \models \alpha \vee \beta$ holds. Other cases can be treated in the same way. It is easy to verify our lemma for the case where φ is of the form $\alpha \wedge \beta$. The case where φ is of the form $\alpha \& \beta$ is treated similarly to the case where φ is $\alpha \vee \beta$.

LEMMA 3.3. *In every strong model* $\langle \boldsymbol{M}, K, \models \rangle$,

(d′) $a \models \alpha \vee \beta$ *if and only if for some* $b, c \in K$ *such that* $a \geqq b \cap c$, $b \models \alpha$ *and* $c \models \beta$.

PROOF. The if part is trivial. Suppose that $a \models \alpha \vee \beta$.

Then, there exist $d, e \in K$ such that $a \geqq d \cap e$, ($d \models \alpha$ or $d \models \beta$) and ($e \models \alpha$ or $e \models \beta$). If either ($d \models \alpha$ and $e \models \beta$) or ($e \models \alpha$ and $d \models \beta$), then the only if part holds. Suppose otherwise. Then, either ($d \models \alpha$ and $e \models \alpha$) or ($d \models \beta$ and $e \models \beta$). In the former case, we have $a \models \alpha$ by Lemma 3.2. By Lemma 3.1, $\infty \models \beta$. Thus, $a \geqq a \cap \infty$, $a \models \alpha$ and $\infty \models \beta$ hold. The latter case can be treated in the same way.

The result similar to Lemma 3.2 holds also for every weak model. We will show this by using the following lemma concerning a certain relationship between strong models and weak ones, instead of proving this directly.

LEMMA 3.4. *Suppose that* $\langle \boldsymbol{M}, K, \models \rangle$ *is a weak model. Let* $\boldsymbol{M}^*$ *be an SO-monoid obtained from* $\boldsymbol{M}$ *by adding a new element* ∞ *with the requirement that* $c \leqq \infty$ *for all* $c \in M$, *and let* K^* *be the set* $K \cup \{\infty\}$. *Then,* $\langle \boldsymbol{M}^*, K^* \rangle$ *is a strong frame. Moreover, if* $\models^*$ *is a strong valuation on* $\langle \boldsymbol{M}^*, K^* \rangle$ *defined by*

$$a \models^* p \text{ if and only if } a \models p$$

for any $a \in K$ *and any propositional variable* p, *then*

$$a \models^* \varphi \text{ if and only if } a \models \varphi$$

for any $a \in K$ *and any formula* φ.

PROOF. To show that $\langle \boldsymbol{M}^*, K^* \rangle$ is a frame, we must prove that K^* satisfies conditions (i) and (ii) for frames. Here we will show only that K^* satisfies (ii). Suppose $a \in K^*$ and $b \cap c \leqq a$ for $b, c \in M^*$. Since b has an extension ∞ in K^*, we have to prove that

(1) there exists $b' \in K^*$ such that $b \leqq b'$ and $b' \cap c \leqq a$.

If either $a = \infty$ or $b = \infty$, then (1) holds if we take ∞ for b'. So we suppose that $a \neq \infty$ and $b \neq \infty$. If $c = \infty$ then $b \cap c = b \leqq a$. So it suffices to take a for b', in this case. If $c \neq \infty$ and b has no extensions in K then $c \leqq a$, and hence we can take ∞ for b'. If $c \neq \infty$ and b has an extension in K, then there exists $b'' \in K$ such that $b \leqq b''$ and $b'' \cap c \leqq a$, since K satisfies (ii). So, (1) holds if we take b'' for b'. The second part of our lemma can be shown by using induction on the length of φ.

COROLLARY 3.5. *Let $\langle \mathbf{M}, K, \models \rangle$ be a weak model. Then for any formula φ and any $a, b, c \in K$,*

$$\text{if } a \models \varphi,\ b \models \varphi \text{ and } a \cap b \leqq c, \text{ then } c \models \varphi.$$

PROOF. Let $\langle \mathbf{M}^*, K^*, \models^* \rangle$ be a strong model obtained from $\langle \mathbf{M}, K, \models \rangle$ by the way mentioned in the previous lemma. By Lemma 3.2,

$$\text{if } a \models^* \varphi,\ b \models^* \varphi \text{ and } a \cap b \leqq c \text{ then } c \models^* \varphi$$

for any formula φ and any $a, b, c \in K$ ($\subseteq K^*$). From this, our corollary follows immediately by using the second part of Lemma 3.4.

COROLLARY 3.6. *If a formula φ is valid in every strong model then φ is valid also in every weak model.*

PROOF. Suppose that for some $a \in K$, $a \not\models \varphi$ holds in a weak model $\langle \mathbf{M}, K, \models \rangle$. Then, $a \not\models^* \varphi$ holds also in the strong model $\langle \mathbf{M}^*, K^*, \models^* \rangle$, by Lemma 3.4. This contradicts our assumption. Hence, φ is valid in every weak model.

We will show a weaker result on the converse of Lemma 3.4.

LEMMA 3.7. *If $\langle \mathbf{M}, M \rangle$ is a total strong frame with the greatest element ∞, then $\langle \mathbf{M}, M - \{\infty\} \rangle$ is a weak frame. Moreover if $\models$ is any strong valuation on $\langle \mathbf{M}, M \rangle$ and $\models'$ is a weak valuation on $\langle \mathbf{M}, M - \{\infty\} \rangle$ defined by*

$$a \models' p \text{ if and only if } a \models p$$

for any $a \in M - \{\infty\}$ and any propositional variable p, then

$$a \models' \varphi \text{ if and only if } a \models \varphi$$

for any $a \in M - \{\infty\}$ and any formula φ.

COROLLARY 3.8. *If a formula φ is valid in every weak model then φ is valid also in every total strong model.*

PROOF. Similar to the proof of Corollary 3.6. In this case, we remark that if $a \not\models \varphi$ holds in a total strong model $\langle \mathbf{M}, M, \models \rangle$ then $a \neq \infty$ and hence $a \in M - \{\infty\}$, since $\infty \models \alpha$ for every formula α.

§4. Completeness theorem. We will prove the completeness theorem for L_{BCC}, L_{BCK} and LJ^* with respect to our semantics introduced in the previous section.

A *commutative* PO-monoid is a PO-monoid in which $ab = ba$ holds for all a, b. An *idempotent* PO-monoid is a PO-monoid in which $aa = a$ holds for all a.

LEMMA 4.1. *Any idempotent PO-monoid is a join-semilattice, in which the join $a \cup b$ of two elements a and b is equal to ab. Hence, any idempotent PO-monoid is commutative, and any idempotent SO-monoid is a lattice ordered monoid.*

PROOF. As remarked before, both $a \leqq ab$ and $b \leqq ab$ hold. So, ab is an upper bound of a and b. Let $a \leqq x$ and $b \leqq x$. Then, $ab \leqq xb \leqq xx = x$ by the idempotency. Thus, ab is the least upper bound of a and b. From this, the second part follows immediately.

A frame is called a *frame for* L_{BCC}. A frame $\langle \boldsymbol{M}, K \rangle$ is called a *frame for* L_{BCK} if $\boldsymbol{M}$ is a commutative SO-monoid, and is called a *frame for* LJ^* if $\boldsymbol{M}$ is an idempotent SO-monoid. A weak (or strong) model $\langle \boldsymbol{M}, K, \models \rangle$ is called a weak (or strong) model for L, if the frame $\langle \boldsymbol{M}, K \rangle$ is for L, where L is any of three logics L_{BCC}, L_{BCK} and LJ^*.

The following lemma can be easily verified.

LEMMA 4.2. *Let $\langle \boldsymbol{M}, K, \models \rangle$ be a weak or strong model. Then, for any formula $\alpha_1, \ldots, \alpha_n, \beta$ and any element $a \in K$, the following three conditions are equivalent;*

(1) $a \models \alpha_1 \supset \cdots \supset \alpha_n \supset \beta$.

(2) $a \models \alpha_1 \,\&\cdots\&\, \alpha_n \supset \beta$.

(3) *For any $a_1, \ldots, a_n$, $b \in K$ such that $a_i \models \alpha_i$ for $i = 1, \ldots, n$ and $aa_1 \cdots a_n \leqq b$, $b \models \beta$.*

Suggested by Lemmas 2.1 and 4.2, we have the following definition of the validity of a given sequent in a model. Let $\langle \boldsymbol{M}, K, \models \rangle$ be a weak or strong model. A sequent $\alpha_1, \ldots, \alpha_n \to \beta$ is *valid in* $\langle \boldsymbol{M}, K, \models \rangle$ if the formula $\alpha_1 \,\&\cdots\&\, \alpha_n \supset \beta$ is valid in it.

LEMMA 4.3. *If a sequent $\Gamma \to \delta$ is provable in L then it is valid in every strong model for L, where L is any of L_{BCC}, L_{BCK} and LJ^*.*

PROOF. Our lemma can be shown in the standard way. So, it suffices to see that every initial sequent is valid and that if the upper sequent(s) of a given rule of inference is valid then the lower sequent is also valid. In the following, we will show this only for the case where the rule of inference under consideration is ($\supset \to$). Suppose that both $\Gamma \to \alpha$ and $\Delta, \beta, \Sigma \to \gamma$ are valid in a given strong model $\langle \boldsymbol{M}, K, \models \rangle$. We assume that Γ consists of $\gamma_1, \ldots, \gamma_k$, Δ consists of $\delta_1, \ldots, \delta_m$, and Σ consists of $\sigma_1, \ldots, \sigma_n$. To show that $\Delta, \alpha \supset \beta, \Gamma, \Sigma \to \gamma$ is valid, we assume moreover

(1) $c_1 \models \delta_1, \ldots, c_m \models \delta_m$,

(2) $b \models \alpha \supset \beta$,

(3) $d_1 \models \gamma_1, \ldots, d_k \models \gamma_k$,

(4) $e_1 \models \sigma_1, \ldots, e_n \models \sigma_n$,

(5) $ac_1 \cdots c_m bd_1 \cdots d_k e_1 \cdots e_n \leqq f$,

where $a, c_1, \ldots, c_m, b, d_1, \ldots, d_k, e_1, \ldots, e_n \in K$. We will show that $f \models \gamma$. By the condition (i) for K, there exists an element $g \in K$ such that

(6) $\underline{bd_1 \cdots d_k} \leqq g$,

(7) $\underline{ac_1 \cdots c_m g e_1 \cdots e_n} \leqq f$.

Using (i) again for (6), there exists an element $d \in K$ such that

(8) $\underline{d_1 \cdots d_k} \leqq d$,

(9) $bd \leqq g$.

Since $1 \cdot (d_1 \cdots d_k) \leqq d$ by (8), there exists $h \in K$ such that

(10) $\underline{h} \cdot \underline{(d_1 \cdots d_k)} \leqq d$.

By (3) and (10),

(11) $d \models \alpha$,

since $\Gamma \to \alpha$ is valid. By (2), (9) and (11),

(12) $g \models \beta$.

By (1), (12), (4) and (7), we have $f \models \gamma$, since $\Delta, \beta, \Sigma \to \gamma$ is valid. Thus, $a \models \Delta, \alpha \supset \beta, \Gamma, \Sigma \to \gamma$ holds for all $a \in K$, by Lemma 4.2(3). The case where the rule under consideration is (exchange) or (contraction) can be treated similarly, by using the commutativity or the idempotency of SO-monoids.

Let L be any of the three logics L_{BCC}, L_{BCK} and LJ^*. Let W be the set of all formulas and $\hat{W}$ be the set of all finite (possibly empty) sequences of formulas. For each $\Gamma \in \hat{W}$, define a subset $[\Gamma]$ of W by

$$[\Gamma] = \{\alpha \in W; \Gamma \to \alpha \text{ is provable in } L\}.$$

Next, define

$$T_L = \{[\Gamma]; \Gamma \in \hat{W}\}.$$

The set W belongs to T_L, since $W = [\perp]$. Thus, T_L becomes a partially ordered set with respect to the set inclusion, whose least element is $[\varnothing]$ and whose greatest element is W, where $\varnothing$ denotes the empty sequence of formulas. For every two elements $[\Gamma]$ and $[\Delta]$ in T_L, define

$$\begin{aligned}[\Gamma] \cdot [\Delta] &= \{\alpha \in W; \text{the sequent } \Gamma, \Delta \to \alpha \text{ is provable in } L\} \\ &= [\Gamma, \Delta],\end{aligned}$$

where Γ, Δ is the sequence of formulas consisting of Γ followed by Δ. In the following, the provability of a sequent $\Gamma \to \delta$ in L is denoted by $L \vdash \Gamma \to \delta$. Sometimes we omit the letter L when no confusion will occur.

LEMMA 4.4. *Let* $\mathbf{T}_L = \langle T_L, \cdot, [\varnothing]; \subseteq \rangle$. *Then* $\mathbf{T}_L$ *is an SO-monoid with the greatest element W.*

PROOF. It can be easily verified that $\langle T_L, \cdot, [\varnothing]; \subseteq \rangle$ is a PO-monoid with the greatest element W. Let Γ be a finite nonempty sequence of formulas $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, define a formula $\Gamma^{\&}$ by

$$\Gamma^{\&} = \alpha_1 \& \alpha_2 \& \cdots \& \alpha_n.$$

When $\Gamma = \varnothing$, define $\Gamma^{\&} = p \supset p$. By using Lemma 2.1, we can show that the intersection $[\Gamma] \cap [\Delta]$ of $[\Gamma]$ and $[\Delta]$ is equal to $[\Gamma^{\&} \vee \Delta^{\&}]$ and therefore belongs to T_L. Thus, $\langle T_L, \subseteq \rangle$ is a meet-semilattice.

Now, we will prove that

$$[\Sigma] \cdot [\Gamma] \cap [\Delta]) = [\Sigma] \cdot [\Gamma] \cap [\Sigma] \cdot [\Delta].$$

It is easy to see that

$$[\Sigma] \cdot ([\Gamma] \cap [\Delta]) \subseteq [\Sigma] \cdot [\Gamma] \cap [\Sigma] \cdot [\Delta],$$

by using the monotonicity of the multiplication. For the converse direction, suppose that a formula α is in $[\Sigma] \cdot [\Gamma] \cap [\Sigma] \cdot [\Delta]$. Then, both $\Sigma, \Gamma \to \alpha$ and $\Sigma, \Delta \to \alpha$ are provable. So, $\Sigma, \Gamma^{\&} \to \alpha$ and $\Sigma, \Delta^{\&} \to \alpha$ are also provable, by applying $(\& \to)$ repeatedly. Thus $\Sigma, \Gamma^{\&} \vee \Delta^{\&} \to \alpha$ is provable by $(\vee \to)$. This means that $\alpha \in [\Sigma] \cdot ([\Gamma] \cap [\Delta])$. Similarly, we can show that

$$([\Gamma] \cap [\Delta]) \cdot [\Sigma] = [\Gamma] \cdot [\Sigma] \cap [\Delta] \cdot [\Sigma].$$

The following lemma can be easily shown.

LEMMA 4.5. 1) $\boldsymbol{T}_L$ *is commutative if* $L = L_{\mathrm{BCK}}$.

2) T_L *is idempotent if* $L = LJ^*$.

By these two lemmas, $\langle \boldsymbol{T}_L, T_L \rangle$ has been shown to be a total strong frame for L. This frame can be considered as a kind of *articulated model*, as introduced in Fine [2]. Next, define a weak valuation $\models$ on $\langle \boldsymbol{T}_L, T_L \rangle$ by

$$[\Gamma] \models p \text{ if and only if } p \in [\Gamma], \text{ i.e. } \vdash \Gamma \to p,$$

for any $[\Gamma] \in T_L$ and any propositional variable p. Clearly, $\models$ satisfies conditions (a) and (a′) of strong valuations in §3.

LEMMA 4.6. *Let* $\models$ *be the strong valuation defined just above. Then, for any* $[\Gamma] \in T_L$ *and any formula* φ,

$$[\Gamma] \models \varphi \textit{ if and only if } \varphi \in [\Gamma].$$

PROOF. It suffices to show the following, each of which corresponds to the same-lettered condition of strong valuations. (Recall here that $\langle \boldsymbol{T}_L, T_L \rangle$ is total.)

(b′) $\vdash \Gamma \to \bot$ if and only if $[\Gamma] = W$.

(c) $\vdash \Gamma \to \alpha \supset \beta$ if and only if, for any $\Delta \in \hat{W}$, $\vdash \Delta \to \alpha$ implies $\vdash \Gamma, \Delta \to \beta$.

(d′) $\vdash \Gamma \to \alpha \vee \beta$ if and only if for some $\Delta_1, \Delta_2 \in \hat{W}$ such that $[\Delta_1] \cap [\Delta_2] \subseteq [\Gamma]$, $\vdash \Delta_1 \to \alpha$ and $\vdash \Delta_2 \to \beta$.

(e) $\vdash \Gamma \to \alpha \wedge \beta$ if and only if $\vdash \Gamma \to \alpha$ and $\vdash \Gamma \to \beta$.

(f) $\vdash \Gamma \to \alpha \,\&\, \beta$ if and only if for some $\Delta_1, \Delta_2 \in \hat{W}$ such that $[\Delta_1] \cdot [\Delta_2] \subseteq [\Gamma]$, $\vdash \Delta_1 \to \alpha$ and $\vdash \Delta_2 \to \beta$.

We will give a proof only for (d′). Suppose that $\vdash \Gamma \to \alpha \vee \beta$. If we take α for Δ_1 and β for Δ_2, then $[\Delta_1] \cap [\Delta_2] \subseteq [\Gamma]$, $\vdash \Delta_1 \to \alpha$ and $\vdash \Delta_2 \to \beta$ hold. Conversely, suppose that $\vdash \Delta_1 \to \alpha$ and $\vdash \Delta_2 \to \beta$ for some Δ_1, Δ_2 such that $[\Delta_1] \cap [\Delta_2] \subseteq [\Gamma]$. Then $\vdash \Delta_1{}^{\&} \to \alpha$, $\vdash \Delta_2{}^{\&} \to \beta$ and $\vdash \Gamma \to \Delta_1{}^{\&} \vee \Delta_2{}^{\&}$. Hence, $\vdash \Gamma \to \alpha \vee \beta$ holds. Other cases can be treated in the same way as the above.

COROLLARY 4.7. *If a sequent* $\alpha_1, \alpha_2, \ldots, \alpha_n \to \beta$ *is not provable in L then it is not valid in a total strong model* $\langle \boldsymbol{T}_L, T_L, \models \rangle$.

PROOF. If $\alpha_1, \alpha_2, \ldots, \alpha_n \to \beta$ is not provable then $[\alpha_1, \alpha_2, \ldots, \alpha_n] \not\models \beta$ by Lemma 4.6. Since $[\alpha_i] \models \alpha_i$ for $i = 1, 2, \ldots, n$ and $[\alpha_1] \cdot [\alpha_2] \cdot \cdots \cdot [\alpha_n] = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, it follows that $[\varnothing] \not\models \alpha_1 \,\&\, \alpha_2 \,\&\, \cdots \,\&\, \alpha_n \supset \beta$ by Lemma 4.2. Hence, the sequent $\alpha_1, \alpha_2, \ldots, \alpha_n \to \beta$ is not valid in $\langle \boldsymbol{T}_L, T_L, \models \rangle$.

By Lemma 4.3 and Corollaries 3.6, 3.8 and 4.7, we have the following completeness theorem.

THEOREM 4.8. *Let L be any of three logics* L_{BCC}, L_{BCK} *and* LJ^*, *and* $\Gamma \to \delta$ *be any sequent. Then the following four conditions are equivalent*;

(1) $\Gamma \to \delta$ *is provable in L.*

(2) $\Gamma \to \delta$ *is valid in every strong model for L.*

(3) $\Gamma \to \delta$ *is valid in every weak model for L.*

(4) $\Gamma \to \delta$ *is valid in every total strong model for L.*

So far we have used SO-monoids for our semantics. The existence of meets in SO-monoids is required in order to give an interpretation of formulas containing the disjunction $\vee$. But, for C-fragments, where C does not contain $\vee$, it is not necessary to use SO-monoids. In this case, we can define frames by using PO-monoids and show the completeness theorem, in the same way as the above. In such a case, a weak valuation $\models$ can be defined as a relation satisfying

(a*) $\qquad$ if $a \models p$ and $a \leqq b$ then $b \models p$

for every $a, b \in K$ and every propositional variable p, instead of (a).

§5. Distributive logics. It is well known that the following sequents, each of which means distributivity with respect to $\wedge$ and $\vee$, are provable in the intuitionistic logic LJ:

(1) $$\alpha \wedge (\beta \vee \gamma) \rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma),$$

(2) $$(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \rightarrow \alpha \wedge (\beta \vee \gamma),$$

(3) $$\alpha \vee (\beta \wedge \gamma) \rightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma),$$

(4) $$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \rightarrow \alpha \vee (\beta \wedge \gamma).$$

On the other hand, neither (1) nor (4) is provable in L_{BCK}, while both (2) and (3) are provable in L_{BCC}. This is a reason why we cannot define the interpretation of disjunctive formulas in such a way that

$$a \models \alpha \vee \beta \text{ if and only if } a \models \alpha \text{ or } a \models \beta,$$

in each model for L_{BCC} or L_{BCK}.

We define the formal system L_{DBCC} (or L_{DBCK}) as the system obtained from L_{BCC} (or L_{BCK}) by adding (1) as a new initial sequent. The following lemma is immediate.

LEMMA 5.1. *The above sequent (4) is provable in* L_{DBCC}.

Thus, we call a logic stronger than L_{DBCC} a *distributive* logic. In this section, we will introduce models for L_{DBCC} and L_{DBCK} for which the completeness theorem holds.

An SO-monoid $\mathbf{M} = \langle M, \cdot, 1; \leqq \rangle$ is said to be *distributive*, if for all $a, b, c \in M$ such that $a \cap b \leqq c$, there exists $d \in M$ such that $a \leqq d$, $c \leqq d$ and $d \cap b \leqq c$ (see [3]). By using Lemma 1 of [3, p. 99], we have the following.

LEMMA 5.2. 1) *Suppose that* $\mathbf{M}$ *is an SO-monoid and* $\langle M, \leqq \rangle$ *is a lattice. If* $\langle M, \leqq \rangle$ *is a distributive lattice, then* $\mathbf{M}$ *is distributive in the above sense.*

2) *An SO-monoid* $\mathbf{M}$ *is distributive if and only if for any* $a, b, c \in M$ *such that* $a \cap b \leqq c$, *there exist* $a', b' \in M$ *such that* $a \leqq a'$, $b \leqq b'$ *and* $a' \cap b' = c$.

A frame $\langle \mathbf{M}, K \rangle$ with a distributive SO-monoid M is called a *distributive frame* (abbreviated to a *D-frame*). A model $\langle \mathbf{M}, K, \models \rangle$ is called a *distributive model* (abbreviated to a *D-model*) if $\langle \mathbf{M}, K \rangle$ is a D-frame.

LEMMA 5.3. *The sequent* $\alpha \wedge (\beta \vee \gamma) \rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ *is valid in every strong D-model. Hence, every sequent which is provable in L is valid in every strong D-model for L, where L is either* L_{DBCC} *or* L_{DBCK}.

PROOF. Let $\langle \mathbf{M}, K, \models \rangle$ be any strong D-model. Suppose that $b \models \alpha \wedge (\beta \vee \gamma)$ and $ab \leqq c$, for $a, b, c \in K$. Then $b \models \alpha$ and $b \models \beta \vee \gamma$ hold. So there exist $d, e \in K$ such that $d \models \beta$, $e \models \gamma$ and $d \cap e \leqq b$. Since $\mathbf{M}$ is distributive, there exists $d' \in M$ such that $d \leqq d'$, $b \leqq d'$ and $d' \cap e \leqq b$. Then, there exists also $d'' \in K$ such that $d' \leqq d''$ and $d'' \cap e \leqq b$, by the condition (ii) for K. Hence

(5) $$d \leqq d'', \quad b \leqq d'' \quad \text{and} \quad d'' \cap e \leqq b.$$

Since $e \cap d'' \leqq b$, we can show as above that there exists $e'' \in K$ such that

(6) $$e \leqq e'', \quad b \leqq e'' \quad \text{and} \quad d'' \cap e'' \leqq b.$$

Since $b \models \alpha$ and $d \models \beta$, $d'' \models \alpha \wedge \beta$ holds by (5). Similarly, $e'' \models \alpha \wedge \gamma$ holds, since $b \models \alpha$, $e \models \gamma$ and (6). Moreover, $d'' \cap e'' \leqq b \leqq ab \leqq c$ holds. Hence,

$$c \models (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

holds. From this, the second part follows immediately.

An element a in a meet-semilattice M is said to be *meet-irreducible* in M if $a = b \cap c$ implies that $a = b$ or $a = c$ for every $b, c \in M$. Let $\langle \boldsymbol{M}, K \rangle$ be a D-frame such that each element of K is meet-irreducible in M. Then, $\langle \boldsymbol{M}, K \rangle$ is called a *standard D-frame* (abbreviated to an *SD-frame*). A D-model $\langle \boldsymbol{M}, K, \models \rangle$ is called a *standard D-model* (abbreviated to an *SD-model*), if $\langle \boldsymbol{M}, K \rangle$ is standard. We can show the following.

LEMMA 5.4. *Let $\langle \boldsymbol{M}, K \rangle$ be an SD-frame and $\models$ be a weak or a strong valuation on it. Then, for any $a \in K$,*

$$a \models \alpha \vee \beta \textit{ if and only if } a \models \alpha \textit{ or } a \models \beta.$$

Similarly to Corollary 3.6, we have the following.

LEMMA 5.5. 1) *If a formula φ is valid in every strong D-model then it is valid in every weak D-model.*

2) *If a formula φ is valid in every strong SD-model then it is valid in every weak SD-model.*

PROOF. If suffices to show that if $\boldsymbol{M}$ is distributive then $\boldsymbol{M}^*$ defined in Lemma 3.4 is also distributive, and that the greatest element ∞ is meet-irreducible. But these can be easily verified.

In the following, we mean by a *logic*, a set of formulas closed under two kinds of modus ponens ((m.p.1) and (m.p.2)), and substitution. Sometimes, we identify formal systems L_{BCC}, L_{BCK} and LJ^* etc. with the sets of all formulas provable in them, since each of them is obviously a logic in our sense. Thus, for instance, we say that L_{BCC} is a logic. We sometimes say that a formula is *provable* in a logic L if it belongs to the set L.

Let L be any logic stronger than L_{BCC}, i.e., $L_{\mathrm{BCC}} \subseteq L$. A nonempty set T of formulas is an *L-pretheory* (which is called an *intensional theory* in [20]), if

(a) if $\alpha \in T$ and $\alpha \supset \beta$ is provable in L then $\beta \in T$, and

(b) if $\alpha \in T$ and $\beta \in T$ then $\alpha \wedge \beta \in T$.

Similarly, a set T of formulas is an *L-theory*, if

(c) every formula provable in L belongs to T,

(d) if $\alpha \in T$ and $\alpha \supset \beta \in T$ then $\beta \in T$, and

(e) if $\beta \in T$ and $\alpha \supset \beta \supset \gamma \in T$ then $\alpha \supset \gamma \in T$.

We remark here that every L-pretheory satisfies also (c). For, if β is provable in L then $\alpha \supset \beta$ is also provable in L, where α is any formula in T. Hence $\beta \in T$, by (a). In §4, we have defined the set $[\Gamma]$ for each finite sequence Γ of formulas and each logic L, where L is any of L_{BCC}, L_{BCK} and LJ^*. It is easy to see that each $[\Gamma]$ is an L-pretheory. On the other hand, $[\Gamma]$ is not always an L-theory if L is either L_{BCC} or L_{BCK}.

LEMMA 5.6. 1) *Every L-theory is an L-pretheory, for any logic L stronger than* L_{BCC}.

2) *If a logic L is stronger than the intuitionistic logic, then every L-pretheory is an L-theory.*

PROOF. Since 1) can easily be shown, we will give a proof only of 2). First, we will show that (d) holds. Suppose that both α and $\alpha \supset \beta$ belong to T. Then, $\alpha \wedge (\alpha \supset \beta) \in T$ by (b). On the other hand, $(\alpha \wedge (\alpha \supset \beta)) \supset \beta$ is provable in the intuitionistic logic, and therefore is provable in L by our assumption. So, $\beta \in T$ by (a). Next, we will show that (e) holds. Suppose that both β and $\alpha \supset \beta \supset \gamma$ belong to T. Since $(\alpha \supset \beta \supset \gamma) \supset (\beta \supset \alpha \supset \gamma)$ is provable in L, $\beta \supset \alpha \supset \gamma \in T$ by (a). By using (d), which we have just proved, we have that $\alpha \supset \gamma \in T$.

The following lemma is immediate.

LEMMA 5.7. *Let L be a logic stronger than L_{BCC}, and U be any set of formulas. Then the smallest L-pretheory containing U is equal to the set*

$$\tilde{U} = \{\alpha; \beta_1 \wedge \cdots \wedge \beta_n \supset \alpha \textit{ is provable in L for some formulas } \beta_1, \ldots, \beta_n \textit{ in } U\ (n \geqq 0)\}.$$

It is easy to see the set $\tilde{\varnothing}$ is the smallest L-pretheory and is equal to the set of all provable formulas in L. *In the following, we will assume that L is any of three logics L_{DBCC}, L_{DBCK} and LJ**. Let P_L be the set of all L-pretheories. For each T_1, $T_2 \in P_L$, define

$$T_1 \cdot T_2 = \{\alpha; \beta_1 \supset \beta_2 \supset \alpha \text{ is provable in } L \text{ for some } \beta_1 \in T_1 \text{ and } \beta_2 \in T_2\}.$$

Then it is clear that $T_1 \cdot T_2 \in P_L$. Furthermore, for each T_1, T_2 in P_L, the intersection $T_1 \cap T_2$ belongs also to P_L. The corresponding result for relevant logics to the following lemma is obtained as Lemma 2 of §3 in [2] or Lemma 9 in [20].

LEMMA 5.8. *Let $\mathbf{P}_L = \langle P_L, \cdot, \tilde{\varnothing}; \subseteq \rangle$. Then $\mathbf{P}_L$ is a distributive SO-monoid with the greatest element W.*

PROOF. We will show here only that $\mathbf{P}_L$ is distributive. Let T_1, T_2, T_3 be L-pretheories such that $T_2 \cap T_3 \subseteq T_1$. Define the set T by

$$T = \{\alpha; \beta_1 \wedge \beta_2 \supset \alpha \text{ is provable in } L \text{ for some } \beta_1 \in T_1 \text{ and } \beta_2 \in T_2\}.$$

Clearly, T is also an L-pretheory such that $T_1 \subseteq T$ and $T_2 \subseteq T$. So, it suffices to show that $T \cap T_3 \subseteq T_1$. Suppose that $\alpha \in T \cap T_3$. Then, $\alpha \in T_3$, and $\beta_1 \wedge \beta_2 \supset \alpha$ is provable in L for some $\beta_1 \in T_1$ and $\beta_2 \in T_2$. On the other hand, we can show that if $\beta_1 \wedge \beta_2 \supset \alpha$ is provable in L then the formula $(\alpha \vee (\beta_1 \wedge \beta_2)) \supset \alpha$ is also provable in L. Moreover, we can show that

$$((\alpha \vee (\beta_1 \wedge \beta_2)) \supset \alpha) \supset ((\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2)) \supset \alpha$$

is provable in L, by using Lemma 5.1. Thus,

$$((\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2)) \supset \alpha$$

is provable in L. But, $\alpha \vee \beta_1 \in T_1$ and $\alpha \vee \beta_2 \in T_2 \cap T_3 \subseteq T_1$. So, $(\alpha \vee \beta_1) \wedge (\alpha \vee \beta_2) \in T_1$, and therefore $\alpha \in T_1$.

An L-pretheory T is *consistent* if T is a proper subset of the set of formulas W. Also, an L-pretheory T is *prime* if

$$\alpha \vee \beta \in T \text{ if and only if } \alpha \in T \text{ or } \beta \in T$$

for each formula α, β.

THEOREM 5.9. *An L-pretheory T is prime if and only if T is meet-irreducible in P_L.*

PROOF. Let T be a prime L-pretheory. We suppose that $T = T_1 \cap T_2$ for some L-pretheories T_1 and T_2. If $T \neq T_1$ and $T \neq T_2$ then there exist formulas α and β such that $\alpha \in T_1 - T$ and $\beta \in T_2 - T$. Then, $\alpha \vee \beta \in T_1 \cap T_2 = T$. But this contradicts our assumption that T is prime. Conversely, suppose that T is meet-irreducible. If T is not prime then $\alpha \vee \beta \in T$ but $\alpha \notin T$ and $\beta \notin T$ hold for some α and β. Let T_α and T_β be the smallest L-pretheories containing $T \cup \{\alpha\}$ and $T \cup \{\beta\}$, respectively. Clearly, $T \subsetneq T_\alpha$ and $T \subsetneq T_\beta$. We will show that $T_\alpha \cap T_\beta = T$. Obviously $T \subseteq T_\alpha \cap T_\beta$, so it suffices to show that $T_\alpha \cap T_\beta \subseteq T$. Let $\sigma \in T_\alpha \cap T_\beta$. Then, for some $\gamma_1, \dots, \gamma_m$, $\delta_1, \dots, \delta_n \in T$, both $\gamma_1 \wedge \cdots \wedge \gamma_m \wedge \alpha \supset \sigma$ and $\delta_1 \wedge \cdots \wedge \delta_n \wedge \beta \supset \sigma$ are provable in L, by Lemma 5.7. Let $\varepsilon = \gamma_1 \wedge \cdots \wedge \gamma_m \wedge \delta_1 \wedge \cdots \wedge \delta_n$. Then $\varepsilon \wedge \alpha \supset \sigma$ and $\varepsilon \wedge \beta \supset \sigma$ are both provable in L. Thus, $((\varepsilon \wedge \alpha) \vee (\varepsilon \wedge \beta)) \supset \sigma$ and hence $(\varepsilon \wedge (\alpha \vee \beta)) \supset \sigma$ are provable in L, by using the distributive law (1). But $\varepsilon \wedge (\alpha \vee \beta) \in T$ by our assumption, and so $\sigma \in T$. Hence $T_\alpha \cap T_\beta \subseteq T$, and so we have $T = T_\alpha \cap T_\beta$. But this contradicts the fact that $T \subsetneq T_\alpha$ and $T \subsetneq T_\beta$, since T is meet-irreducible.

By using Zorn's lemma, we have the following.

LEMMA 5.10. *Every consistent L-pretheory can be extended to a consistent, prime L-pretheory.*

LEMMA 5.11. *Let $\boldsymbol{M}$ be an SO-monoid and K be a nonempty set of meet-irreducible elements in M.*

1) *If $\boldsymbol{M}$ is distributive then K satisfies the condition* (ii) *for frames.*

2) *If $\boldsymbol{M}$ is idempotent then $\langle \boldsymbol{M}, K\rangle$ is an SD-frame for LJ*.*

PROOF. 1) Suppose that $b \cap c \leqq a$ for $a \in K$. Then, by Lemma 5.2.2) there exist b^*, $c^* \in M$ such that $b \leqq b^*$, $c \leqq c^*$ and $b^* \cap c^* = a$. Since a is meet-irreducible, either $b^* = a$ or $c^* = a$. Hence, $b \leqq a$ or $c \leqq a$ holds. Now suppose that b has an extension $\tilde{b}$ in K. If $b \leqq a$, then $b \cap c \leqq a \cap c \leqq a$. So, (ii) holds, if we take a for b'. If $b \nleqq a$, then $c \leqq a$. In this case, $\tilde{b} \cap c \leqq c \leqq a$. Hence (ii) holds by taking $\tilde{b}$ for b'. Suppose that b has no extensions in K. Then $b \leqq a$ does not hold since $a \in K$, and therefore $c \leqq a$ holds.

2) We will show first that $\boldsymbol{M}$ is distributive. When $\boldsymbol{M}$ is idempotent, $x \cdot y = x \cup y$ by Lemma 4.1. So $y \leqq x$ holds if and only if $xy = x$. Now, if $a \cap b \leqq c$ then

$$c = c \cdot (a \cap b) = ca \cap cb \geqq ca \cap b.$$

Let $d = ca$. Then $a \leqq d$, $c \leqq d$ and $d \cap b \leqq c$ hold. By using 1) of this lemma, it remains to show that K satisfies the condition (i) for frames. Suppose $bdc \leqq a$ holds for $a \in K$. We will show that $bac \leqq a$. Since $b \leqq a$ and $c \leqq a$, $ab = a$ and $ac = a$ hold. Then

$$a \cdot (bac) = (ab) \cdot (ac) = aa = a.$$

Hence $bac \leqq a$ and $d \leqq a$ hold.

Let Q be the set of all consistent, prime L-pretheories. By Lemma 5.10, Q is not empty.

THEOREM 5.12. *The structure $\langle \boldsymbol{P}_L, Q\rangle$ is an SD-frame for L.*

PROOF. By Lemma 5.8, $\boldsymbol{P}_L$ is a distributive SO-monoid. By Theorem 5.9, Q is a set of meet-irreducible elements in P_L. So, Q satisfies the condition (ii) for frames. It remains to show that Q satisfies also the condition (i) for frames. Let $T \in Q$ and $U, S, V \in P_L$. Moreover suppose that $U \cdot S \cdot V \subseteq T$. We will show that there exists a consistent prime L-pretheory S' such that $S \subseteq S'$ and $U \cdot S' \cdot V \subseteq T$. Now let us consider the set $J = \{R; R$ is an L-pretheory such that $S \subseteq R$ and $U \cdot R \cdot V \subseteq T\}$. It is easy to see that J is a nonempty inductive set. So, there exists a maximal element S' in J, by Zorn's lemma. Since $S' \subseteq U \cdot S' \cdot V \subseteq T \in Q$, S' is consistent. We will show that S' is prime. Suppose otherwise. Then, by Theorem 5.9, there exist L-pretheories S_1 and S_2 such that $S' = S_1 \cap S_2$, $S' \subsetneq S_1$ and $S' \subsetneq S_2$. By the maximality of S', $U \cdot S_1 \cdot V \nsubseteq T$ and $U \cdot S_2 \cdot V \nsubseteq T$. So there exist formulas α and β such that $\alpha \in U \cdot S_1 \cdot V - T$ and $\beta \in U \cdot S_2 \cdot V - T$. Therefore,

$$\alpha \vee \beta \in U \cdot S_1 \cdot V \cap U \cdot S_2 \cdot V = U \cdot S' \cdot V \subseteq T.$$

But this contradicts our assumption that T is prime. As in Lemma 4.5, we can show that $\boldsymbol{P}_L$ is commutative if $L = L_{\mathrm{DBCK}}$ and $\boldsymbol{P}_L$ is idempotent if $L = LJ^*$. Thus, we have our theorem.

We define a weak valuation $\models$ on the SD-frame $\langle \boldsymbol{P}_L, Q \rangle$ by

$$T \models p \text{ if and only if } p \in T$$

for every consistent prime L-pretheory T and every propositional variable p. We must verify that

$$\text{if } T_1 \models p,\ T_2 \models p \text{ and } T_1 \cap T_2 \subseteq T_3 \text{ then } T_3 \models p.$$

But this is obvious.

We have the following lemma, which corresponds to Lemma 3 of §3 in [2] and Lemma 14 in [20], in the case of relevant logics.

LEMMA 5.13. *Let $\models$ be the weak valuation defined just above. Then, for every $T \in Q$ and every formula φ,*

$$\mathrm{T} \models \varphi \textit{ if and only if } \varphi \in T.$$

Now we will show the completeness theorem for some distributive logics.

THEOREM 5.14. *Let L be any of three logics L_{DBCC}, L_{DBCK} and LJ^*, and $\Gamma \to \delta$ be any sequent. Then the following five conditions are equivalent*:

(1) *$\Gamma \to \delta$ is provable in L.*
(2) *$\Gamma \to \delta$ is valid in every strong D-model for L.*
(3) *$\Gamma \to \delta$ is valid in every weak D-model for L.*
(4) *$\Gamma \to \delta$ is valid in every strong SD-model for L.*
(5) *$\Gamma \to \delta$ is valid in every weak SD-model for L.*

PROOF. From (1), (2) follows by Lemma 5.3. By Lemma 5.5, (3) (or (5)) follows from (2) (or (4), respectively). It is obvious that (4) (or (5)) is derived from (2) (or (3), respectively). So, it remains to show that (1) follows from (5). Let Γ be $\gamma_1, \ldots, \gamma_m$ and let ψ be the formula $\gamma_1 \& \cdots \& \gamma_m \supset \delta$. Suppose that $\Gamma \to \delta$ is not provable. Then neither is ψ provable. By using Zorn's lemma, we can show that there exists a consistent prime L-pretheory T such that $\psi \notin T$. Then, $T \not\models \psi$ by Lemma 5.13. This implies that $\Gamma \to \delta$ is not valid in a weak SD-model $\langle \boldsymbol{P}_L, Q, \models \rangle$ for L.

Distributive logics L_{DBCC} and L_{DBCK} are properly stronger than L_{BCC} and L_{BCK}, respectively. But their $\{\bot, \supset, \wedge, \&\}$-fragments are equivalent, as shown in the following theorem.

THEOREM 5.15. *If a formula φ containing no $\vee$ is provable in L_{DBCC} (or L_{DBCK}) then it is provable in L_{BCC} (or L_{BCK}, respectively).*

PROOF. Suppose that φ is not provable in L_{BCC}. By Theorem 4.8 and the remark just below Theorem 4.8, φ is not valid in a total strong model $\langle \boldsymbol{M}, M, \models \rangle$, where $\boldsymbol{M}$ is a PO-monoid. We will extend this valuation $\models$ to a relation $\models^*$ between elements of M and formulas containing also $\vee$, by requiring that

$$a \models^* \alpha \vee \beta \text{ if and only if } a \models^* \alpha \text{ or } a \models^* \beta,$$

for any $a \in M$. Notice that $\models^*$ is no longer a strong valuation in our sense. But we can show that any sequent provable in L_{DBCC} is valid in $\langle \boldsymbol{M}, M, \models^* \rangle$. In particular, if φ is provable in L_{DBCC} then φ must be valid in $\langle \boldsymbol{M}, M, \models^* \rangle$. On the other hand, for any formula α containing no $\vee$,

$$a \models \alpha \text{ if and only if } a \models^* \alpha$$

for any $a \in M$. Thus, $b \not\models^* \varphi$ for some $b \in M$. This is a contradiction. Hence, φ is not provable in L_{DBCC}. Quite similarly, we can show that our theorem holds also for L_{DBCK} and L_{BCK}.

§6. Models for the intuitionistic logic. In this section, we will show how our semantics for the intuitionistic logic relates to Kripke's original one (see [14]). In the following, we will call a Kripke's original frame or a model, simply a *Kripke frame* or a *Kripke model*, respectively.

LEMMA 6.1. *In any weak or strong model $\langle \boldsymbol{M}, K, \models \rangle$ for LJ^*,*

1) $a \models \alpha \supset \beta$ if and only if, for any $b \in K$ such that $a \leqq b$, $b \models \alpha$ implies $b \models \beta$; and

2) $a \models \alpha \& \beta$ if and only if $a \models \alpha$ and $a \models \beta$.

PROOF. 1) Let $a \models \alpha \supset \beta$. Suppose that $a \leqq b$, $b \in K$ and $b \models \alpha$. Then $ab \leqq bb = b \in K$. Thus, $b \models \beta$ holds. Conversely, suppose that for any $b \in K$ such that $a \leqq b$, $b \models \alpha$ implies $b \models \beta$. We assume also that $ac \leqq d$ and $c \models \alpha$ for $c, d \in K$. Then, $a \leqq d$ and $c \leqq d$ hold. So, $d \models \alpha$ holds. Hence $d \models \beta$ holds by our assumption.

2) Notice that the only if part holds always. So, suppose $a \models \alpha$ and $a \models \beta$. Then, $aa \leqq a$. Therefore, $a \models \alpha \& \beta$.

In the rest of this section, we will consider only formulas containing no &.

LEMMA 6.2. *In any idempotent SO-monoid, if $b \cap c \leqq a$ holds for a meet-irreducible element a then either $b \leqq a$ or $c \leqq a$.*

PROOF. By Lemma 4.1,

$$a = a \cup (b \cap c) = a \cdot (b \cap c) = ab \cap ac.$$

Since a is meet-irreducible, either $a = ab = a \cup b$ or $a = ac = a \cup c$ holds. Hence, either $b \leqq a$ or $c \leqq a$ holds.

By Lemma 6.2, we can show that in any SD-model $\langle \boldsymbol{M}, K, \models \rangle$ for LJ^*, the condition that for any $a, b, c \in K$

$$\text{if } b \models p, c \models p \text{ and } b \cap c \leqq a \text{ then } a \models p$$

is equivalent to the condition that for any $a, b \in K$

$$\text{if } b \models p \text{ and } b \leqq a \text{ then } a \models p.$$

Thus, every weak valuation on an SD-frame $\langle M, K\rangle$ for LJ^* is considered as a valuation on a Kripke frame $\langle K, \leqq\rangle$, and vice versa. Combining this fact with Lemmas 5.4 and 6.1, we have the following.

THEOREM 6.3. *Let $\langle M, K, \models\rangle$ be a weak SD-model for LJ^*. Then, $\langle K, \leqq, \models\rangle$ is a Kripke model for the intuitionistic logic such that for every $a \in K$ and every formula φ,*

$$a \models \varphi \text{ in } \langle M, K, \models\rangle \text{ if and only if } a \models \varphi \text{ in } \langle K, \leqq, \models\rangle.$$

Next, we will show the converse of Theorem 6.3 also holds. Suppose that a nonempty partially ordered set $\langle K, \leqq\rangle$ is given. Let us say a subset A of K to be *closed* if, for any $x, y \in K$, $x \in A$ and $x \leqq y$ imply $y \in A$. Define $C(K)$ to be the set of all closed subsets of K. It is well known that $C(K)$ forms a Heyting algebra with respect to the union $\cup$ and the intersection $\cap$. A subset F of a Heyting algebra H with the greatest element 1 is called a *filter* of H, when

1) $1 \in F$,
2) if $a \in F$ and $a \leqq b$ then $b \in F$, and
3) if $a \in F$ and $b \in F$ then $a \cap b \in F$.

Let $F(H)$ be the set of all filters of a Heyting algebra H. Then $F(H)$ is a partially ordered set with respect to the set inclusion $\subseteq$, whose least element is $\{1\}$ and whose greatest element is H. Now, let us consider the partially ordered set $F(C(K))$. For each F, $G \in F(C(K))$, define $F \cdot G$ to be the filter generated by the set $F \cup G$, i.e. $F \cdot G = \{X; X \in C(K)$ and $A \cap B \subseteq X$ for some $A \in F$ and $B \in G\}$. We can easily show that $F \cdot G$ really belongs to $F(C(K))$. Similarly, $F \cap G \in F(C(K))$ if both F and G are in $F(C(K))$. The following lemma can be easily verified.

LEMMA 6.4. *The structure $M_K = \langle F(C(K)), \cdot, \{K\}; \subseteq\rangle$ is an idempotent SO-monoid.*

We will define a mapping h from K to $F(C(K))$ by

$$h(a) = \{C; a \in C \text{ and } C \in C(K)\},$$

for each $a \in K$. Clearly, $h(a)$ is a filter of $C(K)$ for each $a \in K$. Let K^* be the range of h. We will show that h is an order-isomorphism from K to K^*. Suppose that $a \leqq b$ for $a, b \in K$. If $C \in h(a)$, then $a \in C$. Since $a \leqq b$, $b \in C$ and therefore $C \in h(b)$. Thus $h(a) \subseteq h(b)$. Conversely, suppose that $h(a) \subseteq h(b)$. Let $C_a = \{x; x \in K$ and $a \leqq x\}$. Then C_a is a closed set belonging to $h(a)$. Hence $C_a \in h(b)$. Thus $b \in C_a$, which means that $a \leqq b$.

LEMMA 6.5. *Every element in K^* is meet-irreducible in $F(C(K))$.*

PROOF. Let $h(a) = F \cap G$ for some F, $G \in F(C(K))$. Suppose first that $G \subseteq h(a)$. Then $G \subseteq h(a) = F \cap G \subseteq G$. Hence $h(a) = G$. Suppose otherwise. Then, for some $B \in G$, $B \notin h(a)$. Let C be any element in F. Then $B \cup C \in F \cap G = h(a)$, i.e. $a \in B \cup C$. So either $a \in B$ or $a \in C$. But since $B \notin h(a)$, $a \in C$, i.e. $C \in h(a)$. From this it follows that $F \subseteq h(a)$, since C is an arbitrary element. So in this case $F = h(a)$ holds, as above.

By Lemmas 6.4, 6.5 and 5.11, we have the following.

THEOREM 6.6. *Let $\langle K, \leqq\rangle$ be a nonempty partially ordered set. Then there exists an SD-frame $\langle M^*, K^*\rangle$ for LJ^* such that an order-isomorphism h from K to K^* exists. Moreover, for each valuation $\models$ on $\langle K, \leqq\rangle$, a weak valuation $\models'$ on $\langle M^*, K^*\rangle$ satisfying the following condition is determined by this isomorphism h, and vice versa:*

for every $a \in K$ *and every formula* φ,

$$a \models \varphi \text{ in } \langle K, \leqq, \models \rangle \text{ if and only if } h(a) \models' \varphi \text{ in } \langle M^*, K^*, \models' \rangle.$$

Thus we can consider each of Kripke's original models as a special case of our models.

§7. Models for some other nonclassical logics. We can give semantics also for some other nonclassical logics, by making use of our idea. By way of example, we will introduce semantics for Łukasiewicz's many-valued logics and for some logics without the weakening rule, comparing them with those developed in previous works.

First, let us consider Łukasiewicz's logics. (For these logics, see, e.g., [1], [10], [19].) For each positive integer m, L_m denotes the set of all formulas valid in the *(m + 1)-valued model of Łukasiewicz.* We can easily show that the formula $(p \supset p \supset q) \supset (p \supset q)$, which corresponds to the contraction rule, does not belong to L_m for $m \geqq 2$. This fact suggests that it will be possible to apply our method to these logics. Let N be the set of all nonnegative integers. Then, $\boldsymbol{N} = \langle N, +, 0; \leqq \rangle$ is a commutative, linearly ordered PO-monoid. We will define $T_m = \{0, 1, \ldots, m\}$. Of course, T_m is an interval $[0, m]$ in N. So, $\langle \boldsymbol{N}, T_m \rangle$ is a frame, as remarked in §3. We can see that each weak model $\langle \boldsymbol{N}, T_m, \models \rangle$ is nothing else but a model introduced in Urquhart [24] (see also Scott [21]). He proved that for each $m \geqq 0$ and each $\{\bot, \supset\}$-formula α, α belongs to L_{m+1} if and only if α is valid in any weak model $\langle \boldsymbol{N}, T_m, \models \rangle$. On the other hand, we can show that the weak frame $\langle \boldsymbol{N}, N \rangle$ can not characterize Łukasiewicz's $\aleph_0$-valued logic L_ω. In fact, it can be proved that the logic determined by $\langle \boldsymbol{N}, N \rangle$ is incomparable with L_ω.

Let S_m be the set $\{0, 1/m, 2/m, \ldots, (m-1)/m, 1\}$ and S_ω be the set of all rationals in the interval $[0, 1]$. We define a binary relation $\oplus$ by $a \oplus b = \min\{1, a + b\}$. Then, for each $m \leqq \omega$, $\boldsymbol{S}_m = \langle S_m, \oplus, 0; \leqq \rangle$ is a commutative PO-monoid with the greatest element 1. So, each $\langle \boldsymbol{S}_m, S_m \rangle$ becomes a total strong frame. We remark here that each strong model $\langle \boldsymbol{S}_\omega, S_\omega, \models \rangle$ is essentially the same as one defined by Fine, which is referred to in [24]. We can show that for each $m < \omega$ and each $\{\bot, \supset\}$-formula α, α belongs to L_m if and only if α is valid in any strong model $\langle \boldsymbol{S}_m, S_m, \models \rangle$. On the other hand, it has been remarked in [21] and [24] that if α is valid in any strong model $\langle \boldsymbol{S}_\omega, S_\omega, \models \rangle$ then α is in L_ω, but the converse does not hold. The following theorem shows a relation between L_ω and the logic determined by $\langle \boldsymbol{S}_\omega, S_\omega \rangle$.

THEOREM 7.1. *For each* $\{\bot, \supset\}$*-formula* α, $\neg\neg\alpha$ *is valid in any strong model* $\langle \boldsymbol{S}_\omega, S_\omega, \models \rangle$ *if and only if* α *belongs to* L_ω.

PROOF. The only if part is trivial. So let us consider the if part. Let $\models$ be any strong valuation on $\langle \boldsymbol{S}_\omega, S_\omega \rangle$. For each $\{\bot, \supset\}$-formula φ, define $U(\varphi) = \{x; x \models \varphi\}$. Clearly, each $U(\varphi)$ is either of the form $\{x; c < x \leqq 1\}$ or of the form $\{x; c \leqq x \leqq 1\}$, for some rational number c. In each case, the infimum of the set $U(\varphi)$ exists and is equal to c. Now, define an *assignment* f of $\aleph_0$-valued model by

$$f(p) = 1 - \inf U(p) \quad \text{for each propositional variable } p.$$

By using induction, we can show that for each $\{\bot, \supset\}$-formula φ,

$$f(\varphi) = 1 - \inf U(\varphi).$$

Now, suppose that $\alpha \in L_\omega$. Then, $f(\alpha) = 1$ and hence $\inf U(\alpha) = 0$. That is, either $U(\alpha) = S_\omega$ or $U(\alpha) = S_\omega - \{0\}$. In either case, we have immediately that $\neg\neg\alpha$ is valid in $\langle S_\omega, S_\omega, \models \rangle$.

In every logic which we have considered so far, the weakening rule

$$\frac{\Gamma, \Delta \to \gamma}{\Gamma, \alpha, \Delta \to \gamma}$$

holds. Next, we will treat logics without the weakening rule. The semantical study of some of these logics has been made exclusively by relevance logicians, e.g., [2], [20] and [23], and these semantics for relevant logics have some resemblances to ours, as pointed in §1. But relevant logics are usually supposed to satisfy the distributive law. On the other hand, we will start here from logics even without the distributive law.

We will call L_{BCC}^- (L_{BCK}^- or LJ^{*-}), the Gentzen-type formal system obtained from L_{BCC} (L_{BCK} or LJ^*, respectively) by eliminating the above weakening rule and then replacing each initial sequent of the form $\perp \to \alpha$ by a new initial sequent

$$\Gamma, \perp, \Delta \to \alpha.$$

A structure $\boldsymbol{M} = \langle M, \cdot, 1; \leqq \rangle$ is said to be an *SO⁻-monoid*, if $\boldsymbol{M}$ satisfies all the conditions for SO-monoids, except the condition that $1 \leqq a$ for all $a \in M$. By using SO⁻-monoids instead of SO-monoids, we can define frames and models similarly as in §3. But in doing so, we must add another condition in the definition of strong frames. Let $\langle \boldsymbol{M}, K \rangle$ be a frame with an SO⁻-monoid $\boldsymbol{M}$ whose greatest element is ∞. We say $\langle \boldsymbol{M}, K \rangle$ is a *strong frame* if $\infty \circ a = a \circ \infty = \infty$ for each $a \in M$. Of course, this condition is satisfied whenever $1 \leqq a$ holds. Every (strong) frame $\langle \boldsymbol{M}, K \rangle$ with an SO⁻-monoid M is called a *(strong) frame for* L_{BCC}^-. A (strong) frame $\langle \boldsymbol{M}, K \rangle$ for L_{BCC}^- is called a *(strong) frame for* L_{BCK}^- if M is commutative, and is a *(strong) frame for* LJ^{*-} if $\boldsymbol{M}$ is commutative and satisfies

(1) $\qquad aa \leqq a \quad \text{for all } a \in M.$

We note that if $1 \leqq a$ holds for each $a \in M$ then $a \leqq aa$ holds and therefore (1) is equivalent to the condition that $\boldsymbol{M}$ is idempotent. A model $\langle \boldsymbol{M}, K, \models \rangle$, where $\boldsymbol{M}$ is an SO⁻-monoid, is called a model for L if $\langle \boldsymbol{M}, K \rangle$ is a frame for L, where L is any of L_{BCC}^-, L_{BCK}^- and LJ^{*-}. A formula φ is said to be valid in a model $\langle \boldsymbol{M}, K, \models \rangle$ for L_{BCC}^- (L_{BCK}^- or LJ^{*-}) if *$a \models \varphi$ for all $a \in K$ such that $1 \leqq a$*. A sequent $\alpha_1, \ldots, \alpha_m \to \beta$ is valid in a model if the formula $\alpha_1 \supset \cdots \supset \alpha_m \supset \beta$ is valid in it. Similarly as in §4, the second author proved the following theorem (see [13]).

THEOREM 7.2. *Let L be any of three logics L_{BCC}^-, L_{BCK}^- and LJ^{*-}, and $\Gamma \to \delta$ be any sequent. Then the following four conditions are equivalent*:

(1) $\Gamma \to \delta$ *is provable in L.*
(2) $\Gamma \to \delta$ *is valid in every strong model for L.*
(3) $\Gamma \to \delta$ *is valid in every weak model for L.*
(4) $\Gamma \to \delta$ *is valid in every total strong model for L.*

This result can be easily extended to distributive logics. A formal system obtained from L_{BCC}^- (L_{BCK}^- or LJ^{*-}) by adding the distributive law as initial sequents (see §5) is called L_{DBCC}^- (L_{DBCK}^- or LJ_{D}^{*-}, respectively). We can define distributive models (or standard distributive models) for these three logics in the same way as in §5, only replacing SO-monoids by SO$^-$-monoids everywhere in the definition. Then, we can show the completeness theorem for L_{DBCC}^-, L_{DBCK}^- and LJ_{D}^{*-}, quite similarly to the proof of Theorem 5.14.

This semantics leads us to semantics for relevant logics as shown below, since they satisfy the distributive law but not the weakening rule. Here, we will remove the logical symbols & and $\perp$ from our original language and then add a new logical connective $\sim$, which denotes the negation in relevant logics. Of course, in the present case, the logical connective $\supset$ means *entailment*. Let LR be the Gentzen-type formal system in this language, which is obtained from LJ_{D}^{*-} by first eliminating initial sequents of the form $\Gamma, \perp, \Delta \to \alpha$ and rules $(\& \to)$ and $(\to \&)$, and then adding the following two rules for $\sim$:

$$\frac{\Gamma, \alpha \to \sim\beta}{\Gamma, \beta \to \sim\alpha}(\sim 1), \qquad \frac{\Gamma \to \sim\sim\alpha}{\Gamma \to \alpha}(\sim 2).$$

Then, we can verify that this system LR is logically equivalent to the system R of *relevant implication*, discussed in [20].

A pair $\langle M, K\rangle$ of an SO$^-$-monoid M and a subset K of M is called a *frame for LR* if 1) M is commutative and satisfies $aa \leqq a$ for each $a \in M$, 2) K satisfies conditions (i) and (ii) for frames in §3, and 3) a unary operation $*$ satisfying the following conditions is defined on K;

(a) $(a^*)^* = a$ for each $a \in K$.

(b) $cb \leqq a$ implies $c(a^*) \leqq b^*$ for each $a, b \in K$ and each $c \in M$.

A valuation $\models$ on a frame $\langle M, K\rangle$ for LR is defined similarly to valuations on standard distributive frames, except that we use the following (2):

(2) $a \models \sim\alpha$ if and only if $a^* \not\models \alpha$.

If $\langle M, K\rangle$ is a frame for LR and $\models$ is a valuation on it, then the triple $\langle M, K, \models\rangle$ is called a *model for LR*. The validity of a formula or a sequent is defined in the same way as for LJ_{D}^{*-}. By modifying the proof of the completeness theorem for LJ_{D}^{*-} slightly, we have the following.

THEOREM 7.3. *For each sequent $\Gamma \to \delta$, $\Gamma \to \delta$ is provable in the relevant system LR if and only if it is valid in any model for LR.*

We have defined a frame $\langle M, K\rangle$ for LR by using an SO$^-$-monoid M. But it is clear that M need not be a semilattice. Thus, instead of an SO$^-$-monoid, we can take a partially ordered monoid satisfying conditions (i) and (ii) for a PO-monoid in §3, with an operation $*$ on K satisfying the above conditions (a) and (b), in order to define a frame for LR. Then the proof of Theorem 7.3 becomes essentially the same one as developed in [2] or [20]. In [20], Routley and Meyer made use of the subset K with the ternary relation R on it, while we take a partially ordered monoid M with its subset K as a basic structure. From our standpoint, the relation $Rabc$ can be considered as the restriction of the relation $ab \leqq c$ (on M) to K (see Lemma 11 of [20]). Fine's approach in [2] is more similar to ours in this respect (see Fine's discussion in 3 of §5 in [2]).

§8. Embedding theorems. As mentioned in §1, our research on logics without the contraction rule is motivated by the study of BCK-algebras. So we will devote this section to applications of our research to the study of BCK-algebras. Here, it should be remarked that Kripke-type semantics and the corresponding algebraic structures are the dual of each other. By taking notice of this duality, we can get embedding theorems for these algebraic structures. As for recent development in the study of BCK-algebras, see Komori [11], [12] and works by the Krakow group [7], [15], [25], [26].

For each logical symbol $\supset$, $\vee$, $\wedge$, &, $\perp$, we will use $\rightarrow$, $\cup$, $\cap$, $*$, 0, respectively, to denote the corresponding algebraic operation or constant. Let C be a subset of $\{\supset, \vee, \wedge, \&, \perp\}$. Then, C^* denotes a subset of $\{\rightarrow, \cup, \cap, *, 0\}$ obtained from C by replacing each logical symbol in C by the corresponding algebraic operation (or constant). Similarly, for each formula φ, define φ^* to be a term obtained from φ by first replacing propositional variables $p, q, r, \ldots$ in φ by distinct variables $x, y, z, \ldots$, respectively, and then replacing each logical symbol in φ by the corresponding algebraic operation (or constant).

An algebra $\boldsymbol{A} = \langle A, \rightarrow, \cup, \cap, *, 0, 1\rangle$ is a *full BCC-algebra* (or a *full BCK-algebra*) if it satisfies the conditions

(a) $\varphi^* = 1$ for each provable formula φ of L_{BCC} (or L_{BCK}),

(b) $1 \rightarrow x = x$, and

(c) if $x \rightarrow y = 1$ and $y \rightarrow x = 1$ then $x = y$.

Similarly, we will define *C^*-fragmentary BCC-algebras* and *C^*-fragmentary BCK-algebras* for each subset C of logical symbols, containing at least $\supset$. They will be sometimes abbreviated to *C^*-BCC-algebras* and *C^*-BCK-algebras*. An algebra $\boldsymbol{A}$ with 1 and operations or constant in C^* is a C^*-BCC- (or C^*-BCK-) algebra, if it satisfies (b), (c) and, instead of (a),

(d) $\varphi^* = 1$ for every provable formula φ of the C-fragment of L_{BCC} (or L_{BCK}).

It can be easily verified that any $\{\rightarrow\}$-BCK-algebra is nothing but a (dual of a) BCK-algebra in the usual sense. By using Corollary 2.8 and considering the *Lindenbaum algebra* of the C-fragment of L_{BCC} or L_{BCK}, we have the following.

THEOREM 8.1. *For each subset C of logical symbols, containing at least $\supset$, the following two conditions are equivalent. For every C-formula φ,*

1) *φ is provable in the C-fragment of H_{BCK},*

2) *$\varphi^* = 1$ always holds in every C^*-BCK-algebra.*

Similar equivalences hold also for the C-fragments of H_{BCC} and C^*-BCC-algebra when either C does not contain $\vee$ or C contains $\wedge$.

This theorem gives us a way of getting axioms of C^*-BCC- or C^*-BCK-algebras. For example, the axioms of full BCK-algebras consist of (b), (c) and

(e) $\varphi' = 1$ for each axiom schema φ of H_{BCK},

where φ' denotes a term obtained from an axiom schema φ presented in §2, by first replacing α, β, γ (see §2) by distinct variables x, y, z, respectively, and then replacing each logical symbol in φ by the corresponding algebraic operation (or constant).

In any C^*-BCC- or C^*-BCK-algebra $\boldsymbol{A}$, we can introduce a relation $\leqq$ by

$$a \leqq b \text{ if and only if } a \rightarrow b = 1.$$

Clearly, $\leqq$ is an order relation on A. Moreover if C contains $\cup$ and $\cap$, then $a \cup b$ and $a \cap b$ are equal to the supremum and the infimum of a and b with respect to the order $\leqq$, and hence $\boldsymbol{A}$ is a lattice.

Let $\boldsymbol{M} = \langle M, \cdot, 0; \leqq \rangle$ be a PO-monoid. Similarly as in §6, we say a subset A of M is *closed*, if for any $x, y \in M$,

(1) $$\text{if } x \in A \text{ and } x \leqq y, \text{ then } y \in A.$$

The set of all closed subsets of M is denoted by $C(M)$. Next, suppose that $\boldsymbol{M}$ is an SO-monoid. Then, a subset A of M is said to be $\cap$-*closed* if A is closed and

(2) $$\text{if } x \in A \text{ and } y \in A, \text{ then } x \cap y \in A.$$

The set of all $\cap$-closed subsets of M is denoted by $D(M)$. Notice that both $\varnothing$ and M are closed and are also $\cap$-closed when $\boldsymbol{M}$ is an SO-monoid. For each subset A, B of M for a PO-monoid $\boldsymbol{M}$, define

(3) $$A \to B = \{x; x \in M \text{ and } x * A \subseteq B\},$$

where $x * A$ denotes the set $\{z; z \in M \text{ and } xy \leqq z \text{ for some } y \in A\}$,

(4) $$A \cap B = \text{the intersection of } A \text{ and } B,$$

(5) $$A * B = \{x; x \in M \text{ and } yz \leqq x \text{ for some } y \in A \text{ and some } z \in B\}.$$

Moreover, for all subsets A, B of M for an SO-monoid $\boldsymbol{M}$, define

(6) $$A \vee B = \{x; x \in M \text{ and } y \cap z \leqq x \text{ for some } y, z \in A \cup B \text{ (the union of } A \text{ and } B)\}.$$

The following lemma can be easily proved.

LEMMA 8.2. *If $\boldsymbol{M}$ is a PO-monoid and $A, B \in C(M)$ then $A \to B$, $A \cap B$ and $A * B$ belong to $C(M)$. Moreover, if $\boldsymbol{M}$ is an SO-monoid and $A, B \in D(M)$ then $A \to B$, $A \cap B$, $A * B$ and $A \vee B$ belong to $D(M)$.*

For each $D \supseteq \{\to, \cap\}$, a D-BCC- or D-BCK-algebra $\boldsymbol{A}$ is said to be $\cap$-*complete*, if the infimum $\bigcap_{i \in I} a_i$ exists for every subset $\{a_i; i \in I\}$ of A. Similarly, for each $D \supseteq \{\to, \cap, \cup\}$, a D-BCC- or D-BCK-algebra A is said to be *complete*, if both the infimum $\bigcap_{i \in I} a_i$ and the supremum $\bigcup_{i \in I} a_i$ exist for every subset $\{a_i; i \in I\}$ of A. A weaker form of the following lemma is proved in [8].

LEMMA 8.3. 1) *If $\boldsymbol{M}$ is a (commutative) PO-monoid then $\langle C(M), \to, \cap, *, \varnothing, M \rangle$ is an $\cap$-complete $\{\to, \cap, *, 0\}$-BCC- (or BCK-) algebra.*

2) *If $\boldsymbol{M}$ is a (commutative) SO-monoid then $\langle D(M), \to, \vee, \cap, *, \varnothing, M \rangle$ is a complete full BCC- (or BCK-) algebra.*

PROOF. We must check first that $\langle C(M), \to, \cap, *, \varnothing, M \rangle$ (or $\langle D(M), \to, \vee, \cap, *, \varnothing, M \rangle$) satisfies all the conditions of $\{\to, \cap, *, 0\}$-fragmentary (or full) BCC- or BCK-algebras. But this can be done quite similarly to the proof of the soundness theorem for L_{BCC} or L_{BCK}, with respect to total models determined by PO- or SO-monoids. So we will omit the proof. So we will show that $\langle D(M), \to, \vee, \cap, *, \varnothing, M \rangle$ is complete for any SO-monoid $\boldsymbol{M}$. Let us suppose that a subset $\{A_i; i \in I\}$ of $D(M)$ is given. The existence of the infimum $\bigcap_{i \in I} A_i$ is trivial. Now, define a subset $\bigvee_{i \in I} A_i$ of M by

$$\bigvee_{i \in I} A_i = \left\{x; x \in M \text{ and } \bigcap_{j=1}^{m} a_j \leqq x \text{ for some } a_1, \ldots, a_m \in \bigcup_{i \in I} A_i\right\}.$$

Clearly, $\bigvee_{i \in I} A_i$ belongs to $D(M)$ and is an upper bound for $\{A_i; i \in I\}$. Now, suppose that B is a set in $D(M)$ which is an upper bound for $\{A_i; i \in I\}$. Let x be any element in $\bigvee_{i \in I} A_i$. Then there exist $a_1, \ldots, a_m \in \bigcup_{i \in I} A_i$ such that $\bigcap_{j=1}^m a_j \leqq x$. Since $a_1, \ldots, a_m \in \bigcup_{i \in I} A_i \subseteq B$, it follows that $\bigcap_{j=1}^m a_j \in B$ and hence $x \in B$. Thus, $\bigvee_{i \in I} A_i \subseteq B$, which means that $\bigvee_{i \in I} A_i$ is the supremum of $\{A_i; i \in I\}$ in $D(M)$. We can prove similarly that $\langle C(M), \rightarrow, \cap, *, \emptyset, M \rangle$ is $\cap$-complete.

The following is an algebraic analogue of Theorem 5.15.

COROLLARY 8.4. *If $\boldsymbol{M}$ is a (commutative) PO-monoid then $\langle C(M), \rightarrow, \cup, \cap, *, \emptyset, M \rangle$ is a distributive complete full BCC- (or BCK-) algebra.*

We can show the following lemma in the same way as the proof of Lemma 5.3.

LEMMA 8.5. *If $\boldsymbol{M}$ is a (commutative) distributive SO-monoid then $\langle D(M), \rightarrow, \vee, \cap, *, \emptyset, M \rangle$ is a distributive complete full BCC- (or BCK-) algebra, i.e. it is distributive as a lattice.*

Let $\boldsymbol{M}$ be an SO-monoid and K be a subset of M. A subset A of K is said to be *weakly $\cap$-closed* in K if for any $x, y \in A$, $x \cap y \leqq z$ and $z \in K$ imply $z \in A$. Clearly, $\cap$-closed subsets of K are weakly $\cap$-closed. The following lemma can be easily verified.

LEMMA 8.6. *Let $\langle M, K \rangle$ be a frame. Then, a subset A of K is weakly $\cap$-closed in K if and only if $A = A' \cap K$ for some $\cap$-closed subset A' of M.*

Let $\boldsymbol{M}$ be an SO-monoid. For all subsets A, B of M, define

(1) $A \rightarrow' B = \{x; x \in K$ and, for every $z \in K$ and every $y \in A$, $xy \leqq z$ implies $z \in B\}$,

(2) $A \cap' B = A \cap B \cap K$,

(3) $A *' B = \{x; x \in K$ and $yz \leqq x$ for some $y \in A$ and some $z \in B\}$,

(4) $A \vee' B = \{x; x \in K$ and $y \cap z \leqq x$ for some $y, z \in A \cup B\}$.

LEMMA 8.7. 1) *Let $\boldsymbol{M}$ be an SO-monoid. If both A and B are weakly $\cap$-closed in K then so are $A \rightarrow' B$, $A \cap' B$, $A *' B$ and $A \vee' B$.*

2) *Let $\langle \boldsymbol{M}, K \rangle$ be a frame for L_{BCC} (or L_{BCK}) and $D_K(M)$ be the set of all weakly $\cap$-closed subsets of K. Then, $\langle D_K(M), \rightarrow', \vee', \cap', *', \emptyset, K \rangle$ is a complete full BCC- (or BCK-) algebra.*

The following theorem shows a relation between two full BCC- or BCK-algebras $\langle D(M), \rightarrow, \vee, \cap, *, \emptyset, M \rangle$ and $\langle D_K(M), \rightarrow', \vee', \cap', *', \emptyset, K \rangle$.

THEOREM 8.8. *Let $\boldsymbol{M}$ be any SO-monoid and K be a nonempty subset of M. Define a mapping h by $h(A) = A \cap K$ for each $\cap$-closed subset A of M. Then, h is an epimorphism from the $\{\cup, \cap, *, 0\}$-reduct of $\langle D(M), \rightarrow, \vee, \cap, *, \emptyset, M \rangle$ to the $\{\cup, \cap, *, 0\}$-reduct of $\langle D_K(M), \rightarrow', \vee', \cap', *', \emptyset, K \rangle$ if and only if K satisfies the conditions* (i) *and* (ii) *for frames in* §3.

PROOF. The if part is easy. So we will give only a proof of the only if part of our theorem. Suppose that $yz \leqq x$ for $x \in K$ and $y, z \in M$. Let $A = \{u; y \leqq u$ and $u \in M\}$ and $B = \{u;\ z \leqq u$ and $u \in M\}$. Clearly, both A and B are $\cap$-closed. By our assumption, $h(A * B) = h(A) *' h(B)$, i.e.

$$(A * B) \cap K = (A \cap K) *' (B \cap K).$$

Since $x \in (A * B) \cap K = (A \cap K) *' (B \cap K)$, there exist $u \in A \cap K$ and $v \in B \cap K$ such that $uv \leqq x$. This implies that there exist $u, v \in K$ such that $y \leqq u$, $z \leqq v$ and $uv \leqq x$. Now, we will show that K satisfies the condition (i). Suppose that $a \in K$ and $bdc \leqq a$. Then, there exist $b', e \in K$ such that $b \leqq b'$, $dc \leqq e$ and $b'e \leqq a$, by the fact which we have proved just above. Using this again, there exist $d', c' \in K$ such that $d \leqq d'$, $c \leqq c'$ and $d'c' \leqq e$. So, there is an element $d' \in K$ such that $d \leqq d'$ and $bd'c \leqq b'd'c' \leqq b'e \leqq a$. Similarly, we can show that K satisfies (ii), by using the assumption that $h(A \vee B) = h(A) \vee' h(B)$ holds for every A and B in $D(M)$.

Notice that even if K satisfies the conditions (i) and (ii), the epimorphism h defined in the above theorem does not necessarily have the property that $h(A \to B) = h(A) \to' h(B)$ for every $A, B \in D(M)$.

REMARK. Suppose that $\boldsymbol{M}$ is a PO-monoid or an SO-monoid with the greatest element ∞. Let $C^-(M) = C(M) - \{\varnothing\}$, $D^-(M) = D(M) - \{\varnothing\}$ and $D_K^-(M) = D_K(M) - \{\varnothing\}$. Then, Lemma 8.2 holds also for $C^-(M)$ and $D^-(M)$. Moreover, *all the results from Lemma* 8.2 *to Lemma* 8.7 *hold when we replace* $C(M)$ *by* $C^-(M)$, $D(M)$ *by* $D^-(M)$, $D_K(M)$ *by* $D_K^-(M)$, *the word "frame" by the word "strong frame" and* $\varnothing$ *by* $\{\infty\}$, *when* $\boldsymbol{M}$ *has the greatest element* ∞. Similarly, *after this replacement Theorem* 8.8 *still holds if we assume furthermore that K contains* ∞.

We have introduced the notions of pretheories and theories in §5. Suggested by them, we can define prefilters and filters as follows. Let D be a subset of $\{\to, \cup, \cap, *, 0\}$ which contains $\to$, and let A be a D-BCC-algebra. Suppose first that $\cap \notin D$. Then a nonempty subset F of A is a *prefilter* of A if

(a) if $a \in F$ and $a \to b = 1$ then $b \in F$.

When $\cap \in D$, a subset F of A is a prefilter of A if it satisfies (a) and

(b) if $a \in F$ and $b \in F$ then $a \cap b \in F$.

Clearly, each prefilter contains 1 and $\{1\}$ is the smallest prefilter of any D-BCC-algebra. Similarly, a subset F of A is a *filter* of a D-BCC-algebra A, if

(c) $1 \in F$,

(d) if $a \in F$ and $a \to b \in F$, then $b \in F$, and

(e) if $b \in F$ and $a \to b \to c \in F$, then $a \to c \in F$.

A result similar to Lemma 5.6 holds between prefilters and filters.

Let A be any D-BCC-algebra and P_A be the set of all prefilters of A. For each $F, G \in P_A$, the intersection $F \cap G$ is also a prefilter. Moreover, if we define $F \cdot G$ by

$$F \cdot G = \{x;\ a \to b \to x = 1 \text{ for some } a \in F \text{ and } b \in G\},$$

then we can show that $F \cdot G$ is also a prefilter. Similarly to Lemma 5.8, we have the following.

LEMMA 8.9. *If A is a full BCC- (or BCK-) algebra then $\boldsymbol{M} = \langle P_A, \cdot, \{1\}; \subseteq\rangle$ is a (commutative) SO-monoid with the greatest element A. Moreover, if A is distributive then $\boldsymbol{M}$ is a distributive SO-monoid.*

Combining Lemma 8.9 with Lemmas 8.3, 8.5 and the Remark below Theorem 8.8, we have the following.

THEOREM 8.10. *Every full BCC- (or BCK-) algebra A can be embedded into a complete full BCC- (or BCK-) algebra $\boldsymbol{B}_A = \langle D^-(P_A), \to, \vee, \cap, *, \{A\}, P_A\rangle$. Moreover, if A is distributive then $\boldsymbol{B}_A$ is also distributive.*

In fact, a mapping h from A to $D^-(P_A)$ defined by

$$h(a) = \{F;\ F \in P_A \text{ and } a \in F\}$$

determines the above embedding. In Lemma 8.9, the assumption that A is a full BCC- or BCK-algebra is not necessary. So, by the same method as in Theorem 8.10, we can prove the embedding theorem for some fragmentary BCC- or BCK-algebras as well. But we shall not go into the details of this, since we can get a stronger embedding theorem by using another method, as shown below.

Suppose that a D-BCC- or a D-BCK-algebra A is given. Let Q_A be the set of all finite nonempty sequences of elements of A. We use Greek letters $\alpha, \beta, \gamma, \ldots$ to denote elements of Q_A. For each $\alpha = (a_1, \ldots, a_m) \in Q_A$ and each $c \in A, \alpha \to c$ denotes the term $a_1 \to a_2 \to \cdots \to a_m \to c$. For each $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_n)$, $\alpha\beta$ denotes the sequence $(a_1, \ldots, a_m, b_1, \ldots, b_n)$. Let R_A be the set of all finite nonempty subsets of Q_A. For each $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ in R_A, we write $\{\alpha_1, \ldots, \alpha_m\} \simeq \{\beta_1, \ldots, \beta_n\}$ when, for each $\sigma \in Q_A$ and each $c \in A$,

$$\sigma \to \alpha_i \to c = 1 \text{ holds for every } i \leqq m$$

if and only if

$$\sigma \to \beta_j \to c = 1 \text{ holds for every } j \leqq n.$$

Clearly, $\simeq$ is an equivalence relation on R_A. Let S_A be the quotient set $R_A/\simeq$. The equivalence class containing an element $\{\alpha_1, \ldots, \alpha_m\}$ is denoted by $\langle\{\alpha_1, \ldots, \alpha_m\}\rangle$ or simply by $\langle \alpha_1, \ldots, \alpha_m \rangle$.

Now we will define a relation $\leqq$ on S_A by the condition that

$$\langle \alpha_1, \ldots, \alpha_m \rangle \leqq \langle \beta_1, \ldots, \beta_n \rangle,$$

if, for each $\sigma \in Q_A$ and each $c \in A$,

$$\sigma \to \alpha_i \to c = 1 \text{ for every } i \leqq m \text{ implies } \sigma \to \beta_j \to c = 1 \text{ for every } j \leqq n.$$

It is easy to verify that $\leqq$ is well-defined and is an order relation on S_A with the least element $\langle(1)\rangle$. When A contains 0, S_A has also the greatest element $\langle(0)\rangle$. Next we define an operation $\cap$ on S_A by

$$\begin{aligned}\langle \alpha_1, \ldots, \alpha_m \rangle \cap \langle \beta_1, \ldots, \beta_n \rangle &= \langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \rangle \\ &= \langle \{\alpha_1, \ldots, \alpha_m\} \cup \{\beta_1, \ldots, \beta_n\} \rangle.\end{aligned}$$

Then, it can be shown that $\cap$ is well-defined and that $\langle \alpha_1, \ldots, \alpha_m \rangle \cap \langle \beta_1, \ldots, \beta_n \rangle$ is the infimum of $\langle \alpha_1, \ldots, \alpha_m \rangle$ and $\langle \beta_1, \ldots, \beta_n \rangle$ with respect to the order $\leqq$.

We will introduce another operation $\cdot$ on S_A by

$$\langle \alpha_1, \ldots, \alpha_m \rangle \cdot \langle \beta_1, \ldots, \beta_n \rangle = \langle \{\alpha_i\beta_j; i = 1, \ldots, m, j = 1, \ldots, n\} \rangle.$$

We will show that $\cdot$ is well-defined. Suppose $\langle \alpha_1, \ldots, \alpha_m \rangle = \langle \gamma_1, \ldots, \gamma_s \rangle$ and $\langle \beta_1, \ldots, \beta_n \rangle = \langle \delta_1, \ldots, \delta_t \rangle$. Let us take arbitrary elements $\sigma \in Q_A$ and $c \in A$. Suppose that $\sigma \to \alpha_i\beta_j \to c = 1$ for each $i \leqq m$ and each $j \leqq n$, i.e. $\sigma \to \alpha_i \to \beta_j \to c = 1$. Then, it follows from our assumption that $\sigma \to \gamma_k \to \beta_j \to c = 1$ for each $k \leqq s$ and each $j \leqq n$, and hence $\sigma \to \gamma_k \to \delta_h \to c = 1$ for each $k \leqq s$ and each $h \leqq t$, which is equivalent to $\sigma \to \gamma_k\delta_h \to c = 1$. The converse direction holds also. Thus,

$$\langle \alpha_1, \ldots, \alpha_m \rangle \cdot \langle \beta_1, \ldots, \beta_n \rangle = \langle \gamma_1, \ldots, \gamma_s \rangle \cdot \langle \delta_1, \ldots, \delta_t \rangle.$$

It is easy to see that $\langle S_A, \cdot, \langle(1)\rangle \rangle$ is a monoid with the identity $\langle(1)\rangle$. Moreover, we can show the following lemma.

LEMMA 8.11. *For each D-BCC- (or D-BCK-) algebra A, $\langle S_A, \cdot, \langle(1)\rangle; \leqq \rangle$ is a (commutative) SO-monoid.*

PROOF. It is easily shown that $\langle S_A, \cdot, \langle(1)\rangle; \leqq\rangle$ is a PO-monoid. So we will show that it is an SO-monoid. Let $\langle\alpha_1,\ldots,\alpha_m\rangle$, $\langle\beta_1,\ldots,\beta_n\rangle$, $\langle\gamma_1,\ldots,\gamma_s\rangle \in S_A$. Then,

$$\begin{aligned}
&\langle\alpha_1,\ldots,\alpha_m\rangle\cdot(\langle\beta_1,\ldots,\beta_n\rangle \cap \langle\gamma_1,\ldots,\gamma_s\rangle)\\
&\quad = \langle\alpha_1,\ldots,\alpha_m\rangle\cdot\langle\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_s\rangle\\
&\quad = \langle\{\alpha_i\beta_j; i = 1,\ldots,m, j = 1,\ldots,n\} \cup \{\alpha_i\gamma_k; i = 1,\ldots,m, k = 1,\ldots,s\}\rangle\\
&\quad = (\langle\alpha_1,\ldots,\alpha_m\rangle\cdot\langle\beta_1,\ldots,\beta_n\rangle) \cap (\langle\alpha_1,\ldots,\alpha_m\rangle\cdot\langle\gamma_1,\ldots,\gamma_s\rangle).
\end{aligned}$$

Similary,

$$\begin{aligned}
&(\langle\beta_1,\ldots,\beta_n\rangle \cap \langle\gamma_1,\ldots,\gamma_s\rangle)\cdot\langle\alpha_1,\ldots,\alpha_m\rangle\\
&\quad = (\langle\beta_1,\ldots,\beta_n\rangle\cdot\langle\alpha_1,\ldots,\alpha_m\rangle) \cap (\langle\gamma_1,\ldots,\gamma_s\rangle\cdot\langle\alpha_1,\ldots,\alpha_m\rangle).
\end{aligned}$$

Hence $\langle S_A, \cdot, \langle(1)\rangle; \leqq\rangle$ is an SO-monoid.

Combining Lemma 8.11 with Lemma 8.3, Corollary 8.4 and the Remark below Theorem 8.8, we have the following.

THEOREM 8.12. *Suppose that* $\boldsymbol{A}$ *is any D-BCC- (or D-BCK-) algebra, where D is a subset of* $\{\to, \cup, \cap, *, 0\}$ *containing* $\to$. *Then* $\boldsymbol{A}$ *can be embedded into a complete full BCC- (or BCK-) algebra, Moreover, when* $\cup \notin D$, *we can take a distributive algebra for the algebra into which* $\boldsymbol{A}$ *is embedded.*

PROOF. Suppose first that $0 \notin D$. When $\cup \notin D$, we take the algebra $\boldsymbol{B} = \langle C(S_A), \to, \cup, \cap, *, \varnothing, S_A\rangle$. Then by Lemma 8.11 and Corollary 8.4, $\boldsymbol{B}$ is a distributive complete full BCC- (or BCK-) algebra. When $\cup \in D$, we take $\boldsymbol{C} = \langle D(S_A), \to, \vee, \cap, *, \varnothing, S_A\rangle$. Then by Lemma 8.11 and Lemma 8.3, $\boldsymbol{C}$ is a complete full BCC- (or BCK-) algebra. So, it suffices to show existence of a monomorphism from A to $C(S_A)$ or $D(S_A)$. We define a mapping f by

$$f(a) = \{\langle\beta_1,\ldots,\beta_n\rangle; \beta_j \to a = 1 \text{ for each } j \leqq n\}.$$

Then, f is well-defined and $f(a) \in D(S_A) \subseteq C(S_A)$. Moreover, $f(1) = S_A$ holds. We will show that f is injective. Suppose that $a \neq b$ for $a, b \in A$. Then, either $a \to b \neq 1$ or $b \to a \neq 1$. Without loss of generality, we can assume that $a \to b \neq 1$. Then $\langle(a)\rangle \in f(a) - f(b)$. Hence, $f(a) \neq f(b)$. So, it remains to show that f is a homomorphism. We will show that

$$f(a \to b) = f(a) \to f(b).$$

Suppose that $\langle\gamma_1,\ldots,\gamma_s\rangle \in f(a \to b)$, i.e. $\gamma_k \to a \to b = 1$ for each $k \leqq s$. Let $\langle\delta_1,\ldots,\delta_t\rangle$ be an arbitrary element of $f(a)$. Then $\delta_h \to a = 1$ for each $h \leqq t$. Hence $\gamma_k\delta_h \to b = \gamma_k \to \delta_h \to b = 1$ for each $k \leqq s$ and each $h \leqq t$. Thus, $\langle\gamma_1,\ldots,\gamma_s\rangle\cdot\langle\delta_1,\ldots,\delta_t\rangle \in f(b)$ for every $\langle\delta_1,\ldots,\delta_t\rangle \in f(a)$. This means that $\langle\gamma_1,\ldots,\gamma_s\rangle * f(a) \subseteq f(b)$. Hence, $\langle\gamma_1,\ldots,\gamma_s\rangle \in f(a) \to f(b)$. Conversely, suppose that $\langle\gamma_1,\ldots,\gamma_s\rangle \in f(a) \to f(b)$. Since $\langle(a)\rangle$ belongs to $f(a)$, $\langle\gamma_1,\ldots,\gamma_s\rangle\cdot\langle(a)\rangle \in f(b)$. Thus, $\gamma_k \to a \to b = 1$ for each $k \leqq s$. Therefore $\langle\gamma_1,\ldots,\gamma_s\rangle \in f(a \to b)$.

Similarly, $f(a \cap b) = f(a) \cap f(b)$ and $f(a * b) = f(a) * f(b)$ hold when D contains $\cap$ or $*$. We will show that $f(a \cup b) = f(a) \vee f(b)$, when D contains $\cup$. Suppose that

$\langle\beta_1,\ldots,\beta_n\rangle \in f(a \cup b)$. Then

(1) $$\beta_j \to a \cup b = 1 \quad \text{for each } j \leqq n.$$

Clearly, $\langle(a)\rangle \in f(a)$ and $\langle(b)\rangle \in f(b)$. So, it suffices to show that

(2) $$\langle(a)\rangle \cap \langle(b)\rangle = \langle(a),(b)\rangle \leqq \langle\beta_1,\ldots,\beta_n\rangle.$$

Let us take arbitrary elements $\sigma \in Q_A$ and $c \in A$. Suppose that $\sigma \to a \to c = 1$ and $\sigma \to b \to c = 1$. Then, $\sigma \to a \cup b \to c = 1$. So by using (1), $\sigma \to \beta_j \to c = 1$ for each $j \leqq n$. Thus, (2) holds. Conversely, suppose that $\langle\beta_1,\ldots,\beta_n\rangle \in f(a) \vee f(b)$. Then there exist $\langle\gamma_1,\ldots,\gamma_s\rangle, \langle\delta_1,\ldots,\delta_t\rangle \in f(a) \cup f(b)$ such that

(3) $$\langle\gamma_1,\ldots,\gamma_s\rangle \cap \langle\delta_1,\ldots,\delta_t\rangle \leqq \langle\beta_1,\ldots,\beta_n\rangle.$$

So, $\gamma_k \to a \cup b = 1$ for each $k \leqq s$ and $\delta_h \to a \cup b = 1$ for each $h \leqq t$. By (3), $\beta_j \to a \cup b = 1$ for each $j \leqq n$. Thus, $\langle\beta_1,\ldots,\beta_n\rangle \in f(a \cup b)$.

When D contains 0, we must take $\boldsymbol{B}' = \langle C^-(S_A), \to, \cup, \cap, *, \{\langle(0)\rangle\}, S_A\rangle$, instead of $\boldsymbol{B}$, and $\boldsymbol{C}' = \langle D^-(S_A), \to, \vee, \cap, *, \{\langle(0)\rangle\}, S_A\rangle$, instead of $\boldsymbol{C}$. Then the mapping f defined above becomes a mapping from A to $D^-(S_A)$, because $\langle(0)\rangle \in f(a)$ holds for every $a \in A$. Now we will show that $f(0) = \{\langle(0)\rangle\}$. Clearly, $\langle(0)\rangle \in f(0)$. Suppose that $\langle\beta_1,\ldots,\beta_n\rangle \in f(0)$, i.e. $\beta_j \to 0 = 1$ for each $j \leqq n$. Let us take arbitrary elements $\sigma \in Q_A$ and $c \in A$. If $\sigma \to 0 \to c = 1$ then $\sigma \to \beta_j \to c = 1$ for each $j \leqq n$. So $\langle(0)\rangle \leqq \langle\beta_1,\ldots,\beta_n\rangle$. Since $\langle(0)\rangle$ is the greatest element of S_A, $\langle\beta_1,\ldots,\beta_n\rangle = \langle(0)\rangle$. Thus, $f(0) = \{\langle(0)\rangle\}$. Therefore f is a monomorphism also in this case.

Theorem 8.12 is proved independently by Izdiak for D-BCK-algebras (personal communication). We have also the following, quite similarly to the above.

THEOREM 8.13. *Every D-fragmentary Heyting algebra can be embedded into a complete Heyting algebra, if D contains* $\to$.

By using Corollary 8.4 and Lemma 8.9, we have another embedding theorem.

THEOREM 8.14. *Every (commutative) PO-monoid can be embedded into a (commutative) distributive SO-monoid with greatest element.*

PROOF. We assume first that a PO-monoid $\boldsymbol{M}$ without greatest element is given. Let $\boldsymbol{A} = \langle C(M), \to, \cup, \cap, *, \varnothing, M\rangle$. Then $\boldsymbol{A}$ is a distributive full BCC- (or BCK-) algebra, by Corollary 8.4. So, $\boldsymbol{M}^* = \langle P_A, \cdot, M; \subseteq\rangle$ is a (commutative) distributive SO-monoid with greatest element $C(M)$. Now, define a mapping k from M to P_A by

$$k(u) = \{B;\, u \in B \text{ and } B \in C(M)\}$$

for each $u \in M$. Then $k(u)$ is a prefilter of $\boldsymbol{A}$. Suppose that $u \leqq v$ for $u, v \in M$. If $B \in k(u)$ then $u \in B$ and B is closed. Hence $v \in B$. Thus $k(u) \subseteq k(v)$. Conversely, suppose that $u \nleqq v$. Let $C_u = \{x;\, x \in M \text{ and } u \leqq x\}$. Then $C_u \in C(M)$ and $u \in C_u$, but $v \notin C_u$. Thus $C_u \in k(u) - k(v)$. Hence $k(u) \nsubseteq k(v)$, and so k is an order-isomorphism. Next we will show that $k(uv) = k(u) \cdot k(v)$. Suppose that $C \in k(uv)$, i.e. $uv \in C$. Since $u \in C_u$ and $v \in C_v$, it suffices to show that

(a) $$C_u \to C_v \to C = M.$$

Let x be any element in M. We assume that $y \in C_u$ and $z \in C_v$. Then $u \leqq y$ and $v \leqq z$ hold, and so $uv \leqq yz \leqq xyz$. Therefore $xyz \in C$ for every $y \in C_u$ and every $z \in C_v$. Thus (a) holds. Conversely, suppose that $C \in k(u) \cdot k(v)$. Then there exist D, $E \in C(M)$ such that $u \in D$, $v \in E$ and $D \to E \to C = M$. So

$u = 1 \cdot u \in 1 \cdot D \subseteq E \to C$ and hence $u \cdot E \subseteq C$. Since $v \in E$, $uv \in C$. Therefore $C \in k(uv)$. Next suppose that a given PO-monoid $\boldsymbol{M}$ has the greatest element ∞. Then, $\boldsymbol{A}^- = \langle C^-(M), \to, \cup, \cap, *, \{\infty\}, M \rangle$ is a distributive complete full BCC- or BCK-algebra, by the Remark just below Theorem 8.8. By using this $\boldsymbol{A}^-$, we can prove our theorem quite similarly to the above.

§9. Concluding remarks. There exist a lot of interesting problems, concerning logics and algebraic structures treated in this paper, which have not been solved. We will finish by listing some of these problems.

1. As shown in §2, the separation theorem holds for H_{BCK}. Is it possible to formalize the logic L_{BCC} in a Hilbert-style formal system for which the separation theorem holds?

2. As shown in §3, we have succeeded in giving Kripke-type semantics for some propositional logics without the contraction rule. What is Kripke-type semantics for predicate logics without the contraction rule? Some attempts will be seen in [13] and [17].

3. It is well known that the intuitionistic propositional logic has the finite model property, i.e., for every formula α not provable in the intuitionistic logic, we can find a finite Heyting algebra (or a finite Kripke frame) in which α is not valid. Does L_{BCC}, L_{BCK}, L_{DBCC} or L_{DBCK} have the finite model property? And what about the fragments of them?

4. Logics between the intuitionistic and the classical logic (or logics stronger than the intuitionistic) are called intermediate (or superintuitionistic, respectively) logics. A lot of work has been done on these logics (see e.g. [6], [16]). On the other hand, super-Łukasiewicz logics, i.e., logics stronger than $\aleph_0$-valued Łukasiewicz's logic L_ω, have been studied comprehensively in [10]. Of course, they are also stronger than L_{BCK}. So, it will be interesting to develop the study of super-L_{BCC} or super-L_{BCK} logics, in the similar manner.

5. 1) Is it possible to characterize $\aleph_0$-valued logic L_ω by a single frame? 2) Give an axiomatization of the logic determined by a strong total frame $\langle \boldsymbol{S}_\omega, S_\omega \rangle$. Similarly, give an axiomatization of the logic determined by the weak frame $\langle \boldsymbol{N}, N \rangle$. (See §7.)

6. Give a Gentzen-type formal system for the distributive logic L_{DBCC} or L_{DBCK}, for which the cut elimination theorem holds.

7. It is shown in [11] and [25] that neither the class of $\{\to\}$-BCC-algebras nor the class of $\{\to\}$-BCK-algebras is a variety. On the other hand, it is proved in [8] that the class of D-BCK-algebras is a variety if D contains either $\cap$ or $\cup$. What about the class of D-BCC-algebras? Especially, does the class of full BCC-algebras form a variety?

Added in Proof. After submitting our paper, we noticed the paper [28] by G. K. Dardžaniá in which he studied a predicate logic similar to our L_{BCK} and proved the cut elimination theorem. Also, Professor S. Tamura informed us that in [29] he obtained the cut elimination theorem for systems connected with some partially ordered algebraic structures, which have apparently close relations with systems treated in the present paper.

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