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THE NUMBER OF PROOFS FOR A BCK-FORMULA

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In this note, we give a necessary and sufficient condition for a BCK-formula to have the unique normal form proof.

We call implicational propositional formulas formulas for short. BCK-formulas are the formulas which are derivable from axioms $B = (a \to b) \to (c \to a) \to c \to b$, $C = (a \to b \to c) \to b \to a \to c$, and $K = a \to b \to a$ by substitution and modus ponens. It is known that the property of being a BCK-formula is decidable (Jaskowski [11, Theorem 6.5], Ben-Yelles [3, Chapter 3, Theorem 3.22], Komori [12, Corollary 6]). The set of BCK-formulas is identical to the set of provable formulas in the natural deduction system with the following two inference rules.

$$\frac{\begin{array}{c}[\gamma]\\ \vdots\\ \delta\end{array}}{\gamma \to \delta}\ (\to I) \qquad \frac{\gamma \to \delta \quad \gamma}{\delta}\ (\to E).$$

Here γ occurs at most once in $(\to I)$. By the formulae-as-types correspondence [10], this set is identical to the set of type-schemes of closed BCK-λ-terms. (See [5].) A BCK-λ-term is a λ-term in which no variable occurs twice. Basic notions concerning the type assignment system can be found [4]. Uniqueness of normal form proofs has been known for balanced formulas. (See [2, 14].) It is related to the coherence theorem in cartesian closed categories. A formula is balanced when no variable occurs more than twice in it. It was shown in [8] that the proofs of balanced formulas are BCK-proofs. Relevantly balanced formulas were defined in [9], and it was proved that such formulas have unique normal form proofs. Balanced formulas are included in the set of relevantly balanced formulas. We show a necessary and sufficient condition for a BCK-formula to have a unique

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normal form proof using the following notion of minimality. The notion of BCK-minimality was introduced by Komori [13]. A formula α is called a *trivial substitution instance* of β iff α is a substitution instance of β and β is a substitution instance of α.

DEFINITION 1. A formula is *BCK-minimal* iff it is a BCK-formula and it is not a nontrivial substitution instance of another BCK-formula. A BCK-formula β is a *minimal formula of* α iff β is BCK-minimal and α is a substitution instance of β.

It is clear that a BCK-minimal formula is a principal type-scheme of a closed BCK-λ-term.

We identify two λ-terms when they are α-convertible. Similarly, two types are identified when one is a trivial substitution instance of the other.

LEMMA 1 ([7]). *If two closed BCK-λ-terms in β-normal form have the same principal type, then they are identical.*

LEMMA 2 ([8]). *A BCK-formula is BCK-minimal iff it is a principal type-scheme of a closed BCK-λ-term in $\beta\eta$-normal form.*

THEOREM 1. *Given a BCK-formula α, the number of closed BCK-λ-terms in $\beta\eta$-normal form which has type α is identical to the number of minimal formulas of α.*

PROOF. Let α be a BCK-formula. We denote by proof(α) the set of closed BCK-λ-terms in $\beta\eta$-normal form which have type α and we denote by min(α) the set of minimal formulas of α. We define a function from proof(α) to min(α) and show that it is surjective and injective. Let $M \in \text{proof}(\alpha)$. Then M has type α. By the principal type-scheme theorem (Theorem 15.26 of [4]), M has a principal type-scheme. We denote it by pts(M). Since M is in $\beta\eta$-normal form, pts(M) is minimal by Lemma 2. So we have $\text{pts}(M) \in \min(\alpha)$. Thus pts is a function from proof(α) to min(α). Injectivity of pts is immediate from Lemma 1. To prove the surjectivity, let $\beta \in \min(\alpha)$ and apply Lemma 2 to β. Then there is a closed BCK-λ-term N in $\beta\eta$-normal form whose principal type-scheme is β. Therefore pts is surjective. □

One consequence of the theorem is that a BCK-formula α has only a finite number of normal form proofs. In fact, we can enumerate all the minimal formulas instead of λ-terms. Given a formula γ, we denote by $s_0(\gamma)$ the set of formulas β such that γ is a substitution instance of β. Since we identify trivial substitution instances, the set $s_0(\gamma)$ is finite. Next we denote by $s(\gamma)$ the set of BCK-formulas in $s_0(\gamma)$. Since BCK-provability is decidable, we can enumerate the elements of $s(\gamma)$ from $s_0(\gamma)$. Finally note that β is BCK-minimal iff $s(\beta) = \{\beta\}$. Therefore we have

$$\min(\alpha) = \{\beta \in s_0(\alpha) \mid s(\beta) = \{\beta\}\}.$$

Thus we can enumerate all the elements of min(α).

Akama [1] showed that the number of cut-free proof (in sequent calculus) for a BCK-formula is finite.

COROLLARY 1. *A BCK-formula has a unique proof in $\beta\eta$-normal form iff it has a unique minimal formula.*

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