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Predicate Logics without the Structure Rules

Dedicated to the memory of the late Professor Hidetosi Takahasi

Abstract. In our previous paper [5], we have studied Kripke-type semantics for propositional logics without the contraction rule. In this paper, we will extend our argument to predicate logics without the structure rules. Similarly to the propositional case, we can not carry out Henkin's construction in the predicate case. Besides, there exists a difficulty that the rules of inference ($\to\forall$) and ($\exists\to$) are not always valid in our semantics. So, we have to introduce a notion of normal models.

In this paper, we will introduce five predicate logics without the some structure rules (except *LJ*), *LBCA*, *LBCB*, *LBCC*, *LBCK*, and *LJ*. For each of them, the cut elimination theorem holds. Then, we will introduce a Kripke-type semantics with varying domain (a semantics with constant domain has been given by H. Ono [4]). In the proofs of the completeness theorems for those logics (except *LJ*), we can not construct Henkin's theory. It is closely related with the fact that neither the sequent $A\wedge(B\vee C)\to(A\wedge B)\vee(A\wedge C)$ nor the sequent $A\wedge\exists xB(x)\to\exists x(A\wedge B(x))$ can be proved in those logics without the contraction rule. So, we must prove the completeness theorems for those logics without using Henkin's construction by changing the interpretation of $\vee$ and $\exists$. We assume the familiarity with [3] and [5].

1. Syntactical analysis

In the following, we fix a language $\mathscr{L}$ for predicate logics. The language $\mathscr{L}$ contains $\perp$, $\top$ (truth), $\supset$, $\vee$, $\wedge$, &, $\forall$ and $\exists$ as logical connectives. We suppose that $\mathscr{L}$ contains neither function symbols nor individual constants. We remark here that it does not necessarily hold in *LBCA* that $A\to\top$ is provable even if $\to A$ is provable. On the other hand, in *LBCA*, $\top\to A$ is also provable if $\to A$ is provable. In any system with the weakening rule, both sequents $A\to\top$ and $\top\to A$ are provable if $\to A$ is provable.

Now we will introduce a basic logical system called *LBCA*. Roughly speaking, *LBCA* is a formal system obtained from Gentzen's system *LJ* for intuitionistic predicate logic, by eliminating all the structure rules.

Initial sequents of *LBCA* are either of the form

$$\Gamma,\ \perp,\ \Delta\to A$$

for any finite sequences of formulas Γ and Δ and for any formula A, or of the form

$$A \to A$$

for any atomic formula A, or of the form

$$\to \top .$$

Rules of inferences of *LBCA* are as follows;

$$\frac{\Gamma, \Delta \to C}{\Gamma, \top, \Delta \to C} \ (\top\text{-weakening}) \qquad \frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta, \Gamma, \Sigma \to C} \ (\text{cut})$$

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B} \ (\to \supset) \qquad \frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, A \supset B, \Gamma, \Sigma \to C} \ (\supset \to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B} \ (\to \vee 1) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B} \ (\to \vee 2)$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C} \ (\vee \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B} \ (\to \wedge)$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \ (\wedge \to 1) \qquad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} \ (\wedge \to 2)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \& B} \ (\to \&) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \& B, \Delta \to C} \ (\& \to)$$

$$\frac{\Gamma \to A(y)}{\Gamma \to \forall x A(x)} \ (\to \forall) \qquad \frac{\Gamma, A(y), \Delta \to C}{\Gamma, \forall x A(x), \Delta \to C} \ (\forall \to)$$

$$\frac{\Gamma \to A(y)}{\Gamma \to \exists x A(x)} \ (\to \exists) \qquad \frac{\Gamma, A(y), \Delta \to C}{\Gamma, \exists x A(x), \Delta \to C} \ (\exists \to).$$

Here, both $(\to \forall)$ and $(\exists \to)$ must satisfy the eigenvariable condition, i.e., the variable y must not occur in the lower sequent in these inferences. Next, we will take the following three rules;

$$\frac{\Gamma, \Delta \to C}{\Gamma, A, \Delta \to C} \ (\text{weakening}) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C} \ (\text{exchange})$$

$$\frac{\Gamma, A, A, \Delta \to C}{\Gamma, A, \Delta \to C} \ (\text{contraction}).$$

The formal system *LBCC* (or *LBCB*) is obtained from *LBCA* by adding the weakening rule (or the exchange rule, respectively). The system *LBCK* is obtained from *LBCC* by adding the exchange rule. The system *LJ* is obtained from *LBCK* by adding the contraction rule.

In the same way as the propositional case (cf. Theorem 2.3 of [5]), we can prove the cut elimination theorem.

THEOREM 1.1. *The cut elimination theorem holds for LBCA, LBCB, LBCC, LBCK and LJ.*

COROLLARY 1.2. *Craig's interpolation theorem holds for LBCA, LBCB, LBCC, LBCK and LJ. More precisely, if a sequent $\Delta, \Gamma, \Sigma \to C$ is provable in L then there exists a formula A satisfying the following two conditions;*

(1) *$\Gamma \to A$ and $\Delta, A, \Sigma \to C$ are both provable in L,*

(2) *all predicate variables and free variables in A occur both in Γ and $\Delta \cup \Sigma \cup \{C\}$,*

where L is any of LBCA, LBCB, LBCC, LBCK and LJ.

It is well-known that *LJ* is undecidable. On the other hand, owing to lack of the contraction rule, all of these logics except *LJ* are decidable. We shall prove it by the tableau method. It should be mentioned here that some systems lacking the contraction rule have already known to be decidable (see, e.g., [1], [2]).

By a *configuration* we mean a finite collection $\{S1, S2, \ldots, Sn\}$ of sequents. By an *application of the rule R* except the cut rule to the configuration $\{S1, S2, \ldots, Sn\}$ we mean the replacement of this configuration with new one which is like the first except for containing instead of some Si the result (or results) of applying the converse of rule *R* to Si. For example, if $Si = S1 = \Gamma, A \vee B, \Delta \to C$ then the result of applying the rule $(\vee \to)$ to $\{S1, S2, \ldots, Sn\}$ is $\{\Gamma, A, \Delta \to C; \Gamma, B, \Delta \to C; S2; \ldots; Sn\}$ (we sometimes use semicolons instead of commas because commas are used in sequent). By a *tableau* (of *L*) we mean a finite sequence of configurations $C1, C2, \ldots, Cn$ in which each configuration except the first is the result of applying one of the rules (of *L*) to the (not necessarily immediately) preceding configuration. A configuration $\{S1, S2, \ldots, Sn\}$ is *closed* if each Si in it is an initial sequent of *LBCA*. A tableau $C1, C2, \ldots, Cn$ is *closed* if some Ci in it is closed. By a *tableau of L for a sequent S* we mean a tableau $C1, C2, \ldots, Cn$ of *L* in which $C1$ is $\{S\}$. The following is immediate from the cut elimination theorem.

LEMMA 1.3. *A sequent S is provable in L if and only if some tableau of L for S is closed, where L is any of LBCA, LBCB, LBCC, LBCK and LJ.*

We can easily strengthen this lemma. We assume that all free variables are enumerated as $y1, y2, y3, \ldots$. An application of one of the rule $(\forall \to)$ and $(\to \exists)$ is said to be *restricted*, if the free variable y occurs in the lower sequent or y equals $y1$. An application of one of the rule $(\to \forall)$ and $(\exists \to)$ is said to be *restricted*, if the free variable y is the first free variable which does not occur in the lower sequent. An application of the rule other than for quantifiers is always said to be *restricted*. By a *restricted tableau* we mean a tableau $C1, C2, \ldots, Cn$ such that any application of the rule is restricted and $Ci \neq Cj$ for any $i \neq j$.

LEMMA 1.4. *Lemma 1.3 holds even if the word "tableau" is replaced by the phrase "restricted tableau".*

Let S be a sequent. Suppose that the last free variable occurring in S is yn and that the number of occurrences of quantifiers in S is m. *The quantifier degree* of S, denoted by $qd(S)$, is the sum $n+m$. *The quantifier degree of a configuration* C, denoted by $qd(C)$, is defined by $qd(C) = max\{qd(S) \mid S \in C\}$. For our four logics without the contraction rule, we have

LEMMA 1.5. *Let L be any of LBCA, LBCB, LBCC and LBCK. If a configuration C2 is the result of restricted application of some rule of L to C1, then* $qd(C2) \leqslant qd(C1)$.

If $qd(S) \leqslant n$, then no free variable other than $y1, y2, \ldots, yn$ occurs in S. So, no free variable other than $y1, y2, \ldots, yn$ occurs in any sequent in any configuration in a restricted tableau for a sequent S if $qd(S) \leqslant n$. Besides, the number of formulas in any sequent in a tableau for S is equal or less than that of formulas in S. So, we have

LEMMA 1.6. *Only finite number of sequents can occur in a restricted tableau for a sequent S.*

By the above lemmas, we have a decision procedure for our four logics. A restricted tableau which is not closed is called *full* if any application of any rule to any configuration in the tableau can not produce a new configuration which has not yet occured in the tableau.

THEOREM 1.7. *All of predicate logics LBCA, LBCB, LBCC and LBCK are decidable.*

PROOF. We have the following decision procedure. Let a sequent S be given. We continue to construct a restricted tableau for S, until it becomes closed or it becomes full. If it becomes closed (or full), S is provable (or not provable, respectively). This construction is terminable by Lemma 1.6. Q.E.D.

2. Semantics and Soundness Theorem

In this section, we will introduce Kripke-type semantics for our five predicate logics. We will treat only frames called strong frames in [5]. So, we simply call them frames. Besides, in this paper, structures called SO^{-}-monoids and SO-monoids in [5] are called SO-monoids and SO^{+}-monoids, respectively.

DEFINITION 2.1. A structure $\langle M; \cdot, 1, \infty; \leqslant\rangle$ is a *PO-monoid*, if it is a partially ordered monoid having an identity 1 and an infinity ∞ in M; that is,

(1) $\langle M, \cdot, 1\rangle$ is a monoid with the identity 1,

(2) $\langle M, \leqslant\rangle$ is a partially ordered set satisfying $a \leqslant b$ implies $a \cdot c \leqslant b \cdot c$ and $c \cdot a \leqslant c \cdot b$ for all $a, b, c \in M$, and

(3) as for the infinity, the following hold for any $a \in M$; (i) $\infty \geqslant 1$, (ii) $a \cdot \infty = \infty \cdot a \geqslant \infty$, (iii) $\infty \leqslant a$ implies $a = a \cdot \infty$, (iv) $a \cdot a \cdot \infty = a \cdot \infty$.

A *PO*-monoid $\langle M; \cdot, 1, \infty; \leqslant\rangle$ is an *SO-monoid*, $\langle M, \leqslant\rangle$ is a meet-semilattice satisfying $b \cdot \infty \cap a \cdot \infty = c \cdot \infty \cap a \cdot \infty$ implies $a \cdot (b \cap c) = a \cdot b \cap a \cdot c$ and $(b \cap c) \cdot a = b \cdot a \cap c \cdot a$ for all $a, b, c \in M$.

Hereafter, the multiplication $a \cdot b$ in a semigroup is sometimes denoted by juxtaposition ab without a dot. We say b is an *extension of* a if $a \leqslant b$.

REMARK. The infinity ∞ is not necessarily the maximum element of M. Structures in [3] did not have the multiplication operator. So, in [3], we had to define ∞ as a mapping on M to M instead of an element of M. If a structure has the multiplication, we can replace $\infty(a)$ with $a \cdot \infty$.

LEMMA 2.2. *For any SO-monoid* $\langle M; \cdot, 1, \infty; \leqslant\rangle$ *and any* $a, b \in M$, *the following hold*; (1) $a \geqslant \infty$ *implies* $a = aa$, (2) $ab\infty = ba\infty$, (3) $(a \cap b)\infty = a\infty \cap b\infty$.

PROOF. We will show only (2). $ab\infty = ab\infty ab\infty = a\infty bab\infty = a\infty ba\infty b\infty \geqslant ba\infty$. In the same way, we have $ba\infty \geqslant ab\infty$. Q.E.D.

DEFINITION 2.3. A triple $\langle M, K, U\rangle$ is a *Kripke frame*, if M is an *SO*-monoid $\langle M; \cdot, 1, \infty; \leqslant\rangle$, K is a nonempty subset of M, U is a function from K to the power set $P(S)$ for some nonempty set S satisfying the following;

(1) $a \geqslant \infty$ implies $a \in K$, for any $a \in M$,

for any $a, b, c \in K$,

(2) $U(a) \neq \emptyset$,

(3) $a \cap b \leqslant c$ implies $U(a) \cap U(b) \subset U(c)$,

(4) $ab \leqslant c$ implies $U(a) \cup U(b) \subset U(c)$,

for any $a, b, c \in M$ and any $d \in K$,

(5) if $a \cap b \leqslant d$ then a has an extension a' in K such that $a' \cap b \leqslant d$,

(6) if $abc \leqslant d$ then b has an extension b' in K such that $ab'c \leqslant d$ and $U(b') = U(d)$,

(7) if $a \cap b \leqslant d$ and $a\infty = b\infty$ then both a and b have extensions a' and b' in K, respectively, such that $a' \cap b' \leqslant d$ and $U(a') \cap U(b') = U(d)$.

In the above definition, K and U are called a *frame subset* of M (or M) and a *universe function* of M (or $\langle M, K\rangle$), respectively. If $K = M$, such a Kripke frame is said to be *total*. If $U(a)$ is a finite set for any $a \in K$, such a Kripke frame is said to be *with finite domain*.

We use the same terminology and convention on languages concerning Kripke frames as in [3]. Let a Kripke frame $\langle M, K, U\rangle$ be given and a be an element of K. The language obtained from the language $\mathscr{L}$ by adding all names of elements of $U(a)$ is denoted by $\mathscr{L}(a)$. The language $\bigcup_{a\in K} \mathscr{L}(a)$ is denoted by $\mathscr{L}(\langle M, K, U\rangle)$. The set of all closed formulas of $\mathscr{L}(\langle M, K, U\rangle)$ is denoted by $W(\langle M, K, U\rangle)$ (sometimes, denoted simply by W). $AW(\langle M, K, U\rangle)$ (or simply, AW) denotes the set of all closed atomic formulas of $\mathscr{L}(\langle M, K, U\rangle)$. As in [3], a valuation on $\langle M, K, U\rangle$ is a relation K and $AW(\langle M, K, U\rangle)$.

DEFINITION 2.4. A quadruple $\langle M, K, U, \models\rangle$ is a *Kripke model*, if $\langle M, K, U\rangle$ is a Kripke frame and a subset $\models$ of $K\times AW$ ($(a, A)\in \models$ is denoted by $a\models A$) satisfies the following condition (1).

(1) For any $A\in AW(\langle M, K, U\rangle)$ and any $a, b, c\in K$,
- (1–1) $a\models A$ implies $A\in \mathscr{L}(a)$,
- (1–2) if $a \geqslant \infty$ and $A\in\mathscr{L}(a)$ then $a\models A$,
- (1–3) if $a\models A$ and $b\models A$ and $a\cap b\leqslant c$ then $c\models A$,
- (1–4) $a\models \top$ if and only if $1\leqslant a$,
- (1–5) $a\models \bot$ if and only if $\infty\leqslant a$.

Such a relation $\models$ is called a *valuation* (or a *forcing*) on $\langle M, K, U\rangle$. We extend each valuation $\models$ to a relation between K and $W(\langle M, K, U\rangle)$, inductively as follows. For any $A, B\in W(\langle M, K, U\rangle)$ and any $a\in K$,

(2) $a\models A\supset B \Leftrightarrow \forall b, c\in K$ ($ab\leqslant c$ and $b\models A$ imply $c\models B$) and $A, B\in\mathscr{L}(a)$,

(3) $a\models A\vee B \Leftrightarrow \exists b, c\in K$ ($b\cap c\leqslant a$ and $b\models A$ and $c\models B$ and $B\in\mathscr{L}(b)$ and $A\in\mathscr{L}(c)$),

(4) $a\models A\wedge B\Leftrightarrow a\models A$ and $a\models B$,

(5) $a\models A\& B\Leftrightarrow \exists b, c\in K(bc\leqslant a$ and $b\models A$ and $c\models B)$,

(6) $a\models \forall x A(x)\Leftrightarrow \forall b,\ c\in K\forall u\in U(b)(a\cap c\leqslant b$ and $c\geqslant\infty$ and $\forall xA(x)\in\mathscr{L}(c)$ imply $b\models A(u))$ and $\forall xA(x)\in\mathscr{L}(a)$,

(7) $a\models \exists xA(x)\Leftrightarrow \exists G\subset\subset K\big(\bigcap G\leqslant a$ and $\forall b\in G\exists u\in U(b)$ $(b\models A(u))\big)$.

Here, we mean by $G\subset\subset K$ that G is a finite subset of K.

A closed sequent of $\mathscr{L}(\langle M, K, U\rangle)$ $A1, A2, \ldots, An\rightarrow C$ is valid in $\langle M, K, U, \models\rangle$ if, for any $a, a1, a2, \ldots, an, c\in K$ such that $1\leqslant a$, $A1, A2, \ldots, An$, $C\in\mathscr{L}(a)$, $ai\models Ai$ for $i=1, 2, \ldots, n$ and $a\cdot a1\cdot a2\ldots\cdot an\leqslant c$, $c\models C$. A sequent is valid in $\langle M, K, U, \models\rangle$, if any closed sequent obtained from it by substituting constants of $\mathscr{L}(\langle M, K, U\rangle)$ for free variables is valid in $\langle M, K, U, \models\rangle$. A rule of inference is valid in $\langle M, K, U, \models\rangle$, if the lower sequent of it is valid in $\langle M, K, U, \models\rangle$ whenever all the upper sequent(s) of it is valid in $\langle M, K, U, \models\rangle$. By Lemma 2.2, we can prove the following lemma similarly to Lemma 1.4 in [3].

LEMMA 2.5. *For any* $A \in W(\langle M, K, U\rangle)$ *and* $a, b, c \in K$,
(1) $a \models A$ *implies* $a \in \mathscr{L}(a)$,
(2) $a \geqslant \infty$ *and* $A \in \mathscr{L}(a)$ *imply* $a \models A$,
(3) $a \models A$ *and* $b \models A$ *and* $a \cap b \leqslant c$ *imply* $c \models A$.

PROOF. All are shown by induction on the length of A. We will show only (3). We first remark that $A \in \mathscr{L}(c)$ by (1) and Definition 2.3(5). We have six cases, depending on the outermost logical symbol of A. Here, we will treat only two cases.

(i) The case where A is of the form $B \supset C$. Suppose that $cd \leqslant e$ and $d, e \in K$ and $d \models B$. Then $e \geqslant cd \geqslant (a \cap b)d \geqslant (a \cap b)d'$ where $d' = d \cap (a \cap b)\infty$. And $a\infty \cap d'\infty = b\infty \cap d'\infty = (a \cap b \cap d)\infty$. So, by Definition of SO-monoids, $e \geqslant (a \cap b)d' = ad' \cap bd'$. By Definition 2.3(5), ad' has an extension a' in K such that $e \geqslant a' \cap bd'$. By Definition 2.3(6), d' has an extension d'' in K such that $ad'' \leqslant a'$. Then $d'' \geqslant d \cap (a \cap b)\infty$, $d \models B$ and $(a \cap b)\infty \models B$. So, by the hypothesis of induction, we have $d'' \models B$. By $a \models B \supset C$, $ad'' \leqslant a'$ and $d'' \models B$, we have $a' \models C$. Similarly, we have that there exists $b' \in K$ such that $a' \cap b' \leqslant e$ and $b' \models C$. Hence, by the hypothesis of induction, we have $e \models C$.

(ii) The case where A is of the form $\forall x B(x)$. Suppose that $c \cap c' \leqslant d$, $c' \geqslant \infty$, $u \in U(d)$ and $\forall x B(x) \in \mathscr{L}(c')$. Then $d \geqslant a \cap b \cap c' = (a \cap c' \cap b\infty) \cap (b \cap c' \cap a\infty)$. By $(a \cap c' \cap b\infty)\infty = (b \cap c' \cap a\infty)\infty$ and Definition 2.3(7), both $a \cap c' \cap b\infty$ and $b \cap c' \cap a\infty$ have extensions a' and b' in K, respectively, such that $d \geqslant a' \cap b'$ and $U(a') \cap U(b') = U(d)$. By $a \models \forall x B(x)$, $\forall x B(x) \in \mathscr{L}(c' \cap b\infty)$ and $u \in U(a')$, we have $a' \models B(u)$. Similarly, we can prove that $b' \models B(u)$. Hence, by the hypothesis of induction, we have $d \models B(u)$. Q.E.D.

The following Lemma 2.6 and Lemma 2.7 correspond to Lemma 1.5 and Lemma 1.7 in [3], respectively. They can be proved similarly to their corresponding lemmas.

LEMMA 2.6. $a \models A \vee B$ *if and only if* $\exists b, c \in K\big(b \cap c \leqslant a$ *and* $b \models A$ *and* $c \models B$ *and* $U(b) \cap U(c) = U(a)\big)$ *and* $A, B \in \mathscr{L}(a)$.

LEMMA 2.7 (WEAK SOUNDNESS THEOREM FOR *LBCA*). *Axioms and rules of inference of LBCA except* $(\to\forall)$ *and* $(\exists\to)$ *are valid in every Kripke model.*

An SO-monoid is said to be an SO^{+}*-monoid,* if the identity 1 is the minimum element in it. A *commutative* SO-monoid is an SO-monoid in which $ab = ba$ holds for all a, b. An *idempotent* SO-monoid is an SO-monoid in which $aa = a$ holds for all a.

A Kripke frame is called an *LBCA-Kripke* frame. A Kripke frame $\langle M, K, U\rangle$ is called an *LBCB-Kripke* frame if M is a commutative SO-monoid, and is called an *LBCC-Kripke* frame if M is an SO^{+}-monoid.

An *LBCC*-Kripke frame $\langle \boldsymbol{M}, K, U\rangle$ is called an *LBCK-Kripke* frame if $\boldsymbol{M}$ is commutative, and is called an *LJ-Kripke* frame if $\boldsymbol{M}$ is idempotent.

In the same way as Lemma 4.3 in [5], we have

LEMMA 2.8 (WEAK SOUNDNESS THEOREM). *Axioms and rules of inference of L except* $(\to\forall)$ *and* $(\exists\to)$ *are valid in every L-Kripke model, where L is any of LBCA, LBCB, LBCC, LBCK and LJ.*

As stated in [3], the rules of inference $(\to\forall)$ and $(\exists\to)$ are not valid. We have to introduce the notion of normal models in which the above two rules are valid.

DEFINITION 2.9. A Kripke model $\langle \boldsymbol{M}, K, U, \models\rangle$ ($\boldsymbol{M} = \langle M; \cdot, 1, \infty; \leqslant\rangle$) is *normal* (or a valuation $\models$ on $\langle \boldsymbol{M}, K, U\rangle$ is *normal*), if the following two conditions are satisfied for any $a \in K$ and any $\forall x A(x) \in W(\langle \boldsymbol{M}, K, U\rangle)$;

(1) $\forall b \in K \forall u \in U(b)(a \leqslant b$ implies $b \models A(u))$ implies $a \models \forall x A(x)$,

(2) $a \models \exists x A(x)$ implies $\exists G \subset\subset K\big[\bigcap G \leqslant a$ and $\forall b \in G\big(U(a) \subset U(b)$ and $\exists u \in U(b)(b \models A(u))\big)\big]$.

The following lemma corresponding to Lemma 1.9 in [3] is obvious.

LEMMA 2.10. *If a Kripke model is normal, then for any* $a \in K$ *and any* $\forall x A(x) \in W(\langle \boldsymbol{M}, K, U\rangle)$;

(1) $a \models \forall x A(x) \Leftrightarrow \forall b \in K \forall u \in U(b)\big(a \leqslant b$ implies $b \models A(u)\big)$

(2) $a \models \exists x A(x) \Leftrightarrow \exists G \subset\subset K\big[\bigcap G \leqslant a$ and $\forall b \in G\big(U(a) \subset U(b)$ and $\exists u \in U(b)(b \models A(u))\big)\big]$.

Now, we can prove Soundness Theorem quite similarly to Theorem 1.10 in [3].

THEOREM 2.11 (SOUNDNESS THEOREM). *If a sequent* $\Gamma \to A$ *is provable in L, then* $\Gamma \to A$ *is valid in every normal L-Kripke model, where L is any of LBCA, LBCB, LBCC, LBCK and LJ.*

3. Copleteness Theorem

We will prove the completeness theorem for our five predicate logics with respect to our Kripke semantics. We can prove it in the same way as the propositional case in § 4 of [5].

Let L be any of the five logics *LBCA*, *LBCB*, *LBCC*, *LBCK* and *LJ*. Let $U1, U2, \ldots$ be a countable sequence of finite sets of constant symbols such that $\emptyset \neq U1 \subsetneqq U2 \subsetneqq \ldots$. The language obtained form $\mathscr{L}$, which contains neither function symbols nor constant symbols, by adding all constant symbols in Ui will be denoted by $\mathscr{L}(Ui)$. Let Wi be the set of all closed formulas of $\mathscr{L}(Ui)$ and W be the set $\bigcup_{i \in N} Wi$. For each $A \in Wi$ and each $j \geqslant i$, define a subset $[A;j]$ of Wj by

$$[A;j] = \{B \in Wj \mid A \to B \text{ is provable in } L\}.$$

Next, define

$$TL = \{[A;i] \mid A \in Wi \text{ and } i \in N\}.$$

The set Wi belongs to TL, since $Wi = [\bot;i]$. For every two elements $[A;i]$ and $[B;j]$ in TL, define

$$[A;i]\cdot[B;j] = [A\&B; max(i,j)].$$

It is easy to see that the operation $\cdot$ is well-defined.

LEMMA 3.1. *Let* $\boldsymbol{TL} = \langle TL; \cdot, [\top;1]\, W1; \subset \rangle$. *Then* $\boldsymbol{TL}$ *is an SO-monoid.*

PROOF. We can show that $[A;i]\cdot W1 = W1\cdot[A;i] = Wi$ and that $[A;i] \supset W1$ implies $[A;i] = Wi$. By this, we can easily verify that $\boldsymbol{TL}$ is a PO-monoid. Let $i \leqslant j$, $A \in Wi$ and $B \in Wj$. Let all constant symbols in B and not in Ui be $u1, u2, \ldots, un$. We will denote B by $B(u1, u2, \ldots, un)$, where each ui represents all occurrences of ui in B. Let $x1, x2, \ldots, xn$ be variables not occuring in B. Then, we can show that the intersection $[A;i]\cap[B;j]$ is equals to $[A \vee \exists x1\, \exists x2 \ldots \exists xn B(x1, x2, \ldots \ldots, xn); i]$ and therefore belongs to TL. Thus, $\langle TL, \subset\rangle$ is a meet-semilattice.

Now, we will prove that $[B;j]\cdot W1 \cap [A;i]\cdot W1 = [C;k]\cdot W1 \cap [A;i]\cdot \cdot W1$ implies $[A;i]\cdot([B;j]\cap[C;k]) \supset [A;i]\cdot[B;j]\cap[A;i]\cdot[C;k]$. Notice that $min(j,i) = min(k,i)$ holds by our assumption. Suppose that a formula D is in $[A;i]\cdot[B;j]\cap[A;i]\cdot[C;k]$. When $j = k$, the proof is similar to the propositional case. Suppose $j < k$. By $min(j,i) = min(k,i)$, we have $i = min(i,j,k)$. The formula D is in Wj and both $A, B \to D$ and $A, C \to D$ are provable. Let all constant symbols in C and not in Uj be $u1, u2, \ldots, un$. C is denoted by $C(u1, u2, \ldots, un)$. Then the sequent $A, \exists x1 \exists x2 \ldots \exists xn C(x1, x2, \ldots, xn) \to D$ is provable by applying $(\exists\to)$ repeatedly. The formula $\exists x1 \exists x2 \ldots \exists xn C(x1, x2, \ldots, xn)$ belongs to Wj. Thus $A, B \vee \exists x1 \exists x2 \ldots \exists xn C(x1, x2, \ldots, xn) \to D$ is provable by $(\vee\to)$. This means that $D \in [A;i]\cdot([B;j]\cap[C;k])$. Q.E.D.

The following lemma can be easily shown.

LEMMA 3.2. (1) ***TLBCB*** *is commutative.* (2) ***TLBCC*** *is an* SO^+*-monoid.* (3) ***TLBCK*** *is a commutative* SO^+*-monoid.* (4) ***TLJ*** *is an idempotent* SO^+*-monoid.*

Let U be a function from TL to the set $\{Ui \mid i \in N\}$ such that $U([A;i]) = Ui$ for each $[A;i] \in TL$. Then, it is easily shown that the triple $\langle \boldsymbol{TL}, TL, U\rangle$ is a Kripke frame. So, by Lemma 3.2, we have that the triple $\langle \boldsymbol{TL}, TL, U\rangle$ is a total L-Kripke frame. Next, we will define a valuation $\models$ on $\langle \boldsymbol{TL}, TL, U\rangle$ by

$$[A; i] \models B \text{ if and only if } B \in [A; i],$$

for any $[A; i] \in TL$ and any atomic formula $B \in W$. Clearly, satisfies the condition (1) of Definition 2.4.

LEMMA 3.3. *Let* $\models$ *be the valuation defined just above. Then, for any* $[A; i] \in TL$ *and any* $B \in W$,

$$[A; i] \models B \textit{ if and only if } B \in [A; i].$$

PROOF. It suffices to show the following, each of which corresponds to the same-numbered condition of valuations. (Recall that $\langle \boldsymbol{TL}, TL, U\rangle$ is total.)

(2) $B \supset C \in [A; i] \Leftrightarrow \forall [D; j] \in TL(B \in [D; j]$ implies $C \in [A \& D, max(i, j)])$ and $B, C \in Wi$.

(3) $B \vee C \in [A; i] \Leftrightarrow \exists [D; j], [E; k] \in TL$ $([D; j] \cap [E; k] \subset [A; i]$ and $B \in [D; j]$ and $C \in [E; k]$ and $C \in Wj$ and $B \in Wk)$.

(4) $B \wedge C \in [A; i] \Leftrightarrow B \in [A; i]$ and $C \in [A; i]$.

(5) $B \& C \in [A; i] \Leftrightarrow \exists [D; j], [E; k] \in TL$ $([D \& E;\ max(j, k)] \subset [A; i]$ and $B \in [D; j]$ and $C \in [E; k])$.

(6) $\forall x B(x) \in [A; i] \Leftrightarrow \forall [D; j] \in TL \forall k \in N \forall u \in Uj$ $([A; i] \cap Wk \subset [D; j]$ and $\forall x B(x) \in Wk$ implies $B(u) \in [D; j])$ and $\forall x B(x) \in Wi$.

(7) $\exists x B(x) \in [A; i] \Leftrightarrow \exists G \subset \subset TL(\bigcap G \subset [A; i]$ and $\forall [D; j] \in G \exists u \in Uj(B(u) \in [D; j]))$.

We give a proof only for (6). Suppose $\forall x B(x) \in [A; i]$. Then, we have $\forall x B(x) \in Wi$. Suppose that $[A; i] \cap Wk \subset [D; j]$ and $\forall x B(x) \in Wk$. Then, clearly, we have $\forall x B(x) \in [D; j]$. Therefore, $B(u) \in [D: j]$ for any $u \in Uj$. Conversely, suppose that $\forall [D; j] \in TL \forall k \in N \forall u \in Uj([A; i] \cap \cap Wk \subset [D; j]$ and $\forall x B(x) \in Wk$ implies $B(u) \in [D; j])$ and $\forall x B(x) \in Wi$. We will take A for D, i for k, $j > i$ and $u \in Uj - Ui$. Then, we have that $A \to B(u)$ is provable. So, $A \to \forall x B(x)$ is provable by $(\to\forall)$. Thus, we have $\forall x B(x) \in [A; i]$. Q.E.D.

Next, we have to prove that the Kripke model $\langle \boldsymbol{TL}, TL, U, \models\rangle$ is normal.

LEMMA 3.4. *The Kripke model* $\langle \boldsymbol{TL}, TL, U, \models \rangle$ *defined above is normal.*

PROOF. Similarly to the proof of (6) of Lemma 3.3, we can show that it satisfies (1) of Definition 2.9. We will prove only (2) of Definition 2.9. Suppose $\exists x B(x) \in [A; i]$. Let $j > i$, $u \in Uj - Ui$ and $G = \{[B(u); j], Wi\}$. Then, we have that $\bigcap G \subset [A; i]$ and $\forall F \in G\big(Ui \subset U(F)$ and $\exists v \in U(F)$ $(B(v) \in F)\big)$. Q.E.D.

In the same way as Corollary 4.7 in [5], we have

THEOREM 3.5. *If a sequent* $\Gamma \to A$ *is not provable in L then it is not valid in the normal and total L-Kripke model with finite domain* $\langle \boldsymbol{TL}, TL, U, \models \rangle$, *where L is any of LBCA, LBCB, LBCC. LBCK and LJ.*

REMARK. The original Kripke model constructed to show the completeness for *LJ* is not with finite domain. Because Henkin's construction necessarily produce infinite domain.

By Theorem 2.11 and 3.5, we have the completeness theorem.

THEOREM 3.6. *Let L be any of five logics LBCA, LBCB, LBCC, LBCK and LJ, and* $\Gamma \to A$ *be any sequent. Then the following conditions are equivalent;*

(1) $\Gamma \to A$ *is provable in L,*

(2) $\Gamma \to A$ *is valid in every normal L-Kripke model,*

(3) $\Gamma \to A$ *is valid in every normal and total L-Kripke model with finite domain.*

In the propositional case, as in §5 of [5] (also stated in §7 of [5] for *LBCA* and *LBCB*), we can interpret $a \models A \vee B$ as $a \models A$ or $a \models B$ if distributivity with respect to $\wedge$ and $\vee$ holds. But, in the predicate case, we come across some difficulties in interpreting $a \models \exists x A(x)$ as for some $u \in U(a)$ $a \models A(u)$ even if distributivity with respect to $\wedge$ and $\exists$ holds, that is, $A \wedge \exists x B(x) \to \exists x \big(A \wedge B(x)\big)$ holds. The reason is that we can not do Henkin's construction even if both distributivities hold.

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