BCK ALGEBRAS AND LAMBDA CALCULUS

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The name of BCK-algebras originated from three combinators $B \equiv \lambda xyz.x(yz)$, $C \equiv \lambda xyz.xzy$ and $K \equiv \lambda xy.x$ in lambda-calculus. When we give Gentzen type formulation of BCK-algebras, the weakening rule and the exchanging rule correspond to the combinators K and C, respectively. In this note, we will explain those relationship and state exactly the result in Bunder and Meyer [1] (the expression in [1] lacks an accuracy).

The theory of type assignment to lambda-terms is closely related to the implicational fragment of Gentzen's Natural Deduction for intuitionistic logic. Roughly speaking, for any closed lambda-term M and any type assignment α to M, α is a provable implicational formula in intuitionistic logic and M represents the framework of a proof of α in Natural Deduction NJ. If we limit lambda-terms to BCK terms, we obtain the same relationship between BCK terms and BCK logic. It is noted here that any BCK term is stratified though lambda-terms are not necessarily stratified.

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Lambda calculas and type assignment to lambda-terms

We assume the familiarity with the theory of type assignment to lambda-terms (cf. [2]). We will present a list of main facts and open problems on this field.

FACT 1.1. There exists a closed lambda-term M such that $\vdash_{TA_\lambda} M \in \alpha$ if and only if a type-scheme α is provable in intuitionistic logic. Here we regard type-schemes as implicational formulas.

THEOREM 1.2 (Hirokawa [3]). For any lambda-term M: if H is H_w, then

$$M_{H\lambda} \rhd_\beta M.$$

Theorem 1.2 is a positive answer to a problem in the first draft of this note. It strengthens Theorem 9.28(b) in [2].

DEFINITION 1.3. Let α and β be implicational formulas. α sub> β denotes that β is a substitution instance of α. It is obvious that the relation sub> is a pseudo-order relation. Let S be a set of implicational formulas. We call an implicational formula α **minimal** in S, if α is an element of S and α sub> β for any β in S such that β sub> α.

LK (LJ or LBCK) denotes the set of implicational formulas provable in classical logic (intuitionistic logic or BCK logic (cf. [4] or [5]), respectively).

PROBLEM 1.4. For any implicational formula α, are the following (a) and (b) true ? $((((p \supset q) \supset p) \supset p) \supset q) \supset q$

(a) α is minimal in LK if α is minimal in LJ. no.

(b) α is minimal in LJ if α is minimal in LBCK. Yes.

PROBLEM 1.5. Let an implicational formula α be minimal in LJ. Let M and N be closed lambda-terms in $\beta\eta$-normal form. Then, is it true that M is congruent to N if $\vdash_{TA_\lambda} M \in \alpha$ and $\vdash_{TA_\lambda} N \in \alpha$? no: Tatsuta, Mints

If the above problem will be solved in the positive, it means the uniqueness of NJ proof in normal form for a minimal implicational formula in LJ.

2. Type assignment to BCK terms

We begin with the definition of BCK lambda-terms.

DEFINITION 2.1 (BCK lambda-terms). The set of expressions called **BCK lambda-terms**, which is a subset of the set of lambda-terms, is defined inductively as follows:

(a) All variables are BCK lambda-terms.

(b) If M and N are any BCK lambda-terms such that $FV(M) \cap FV(N) = \phi$, then (MN) is a BCK lambda-term.

(c) If M is any BCK lambda-term and x is any variable, then $(\lambda x.M)$ is a BCK lambda-term.

DEFINITION 2.2 (BCK CL-terms). The set of expressions called **BCK CL-terms**, which is a subset of the set of CL-terms, is defined inductively as follows:

(a) All variables and three combinators B, C and K are BCK CL-terms.

(b) If M and N are any BCK CL-terms such that $FV(M) \cap FV(N) = \phi$, then (MN) is a BCK CL-term.

We can define a BCK CL-term λ+x.M for each variable x and each BCK CL-term M, with the property that

$$(\lambda{+}x.M)N \rhd_w [N/x]M.$$

DEFINITION 2.3 (BCK abstraction). For each BCK CL-term M and each x, a BCK CL-term called λ+x.M is defined by induction on M, thus:

(a) $\lambda{+}x.M \equiv KM$ if $x \notin FV(M)$;

(b) $\lambda{+}x.x \equiv I$ where $I \equiv CKK$;

(c) $\lambda{+}x.UV \equiv BU(\lambda{+}x.V)$ if $x \in FV(V)$;

(d) $\lambda{+}x.UV \equiv C(\lambda{+}x.U)V$ if $x \in FV(U)$.

FACT 2.4. $(\lambda{+}x.M)N$ behaves like a β-redex; that is

$$(\lambda{+}x.M)N \rhd_w [N/x]M.$$

DEFINITION 2.5 (The H+-transformation). To each BCK lambda-term M we assosiate a BCK CL-term called M_{H+} (or usually just M_H), thus:

(a) $x_H \equiv x$,

(b) $(MN)_H \equiv M_H N_H$,

(c) $(\lambda x.M)_H \equiv \lambda+x.(M_H)$.

THEOREM 2.6. For any BCK lambda-term M: if H is H+, then

$$M_{H\lambda} =_\beta M.$$

Theorem 2.6 can be strengthened in the same way as Theorem 9.28(b) in [2], as follows.

THEOREM 2.7 (Hirokawa [3]). For any BCK lambda-term M: if H is H+, then

$$M_{H\lambda} \triangleright_\beta M.$$

The following is a BCK version of Fact 1.1.

FACT 2.8. There exists a closed BCK lambda-term M such that $\vdash_{TA_\lambda} M \in \alpha$ if and only if an implicational formula α is provable in BCK logic.

Next, we shall state some results in Bunder and Meyer [1].

FACT 2.9. All BCK lambda-terms are stratified.

We have the following by Fact 2.9 and Theorem 15.26 in [2].

FACT 2.10 (Theorem 1 in [1]). Every closed BCK lambda-term has a principal type-scheme.

FACT 2.11 (An accurate form of Theorem 2 in [1]). If $\alpha \to \beta$ and γ are principal type-scheme of some closed BCK lambda-terms, then there exist substitution instances $\delta \to \beta_1$ and δ of $\alpha \to \beta$ and γ respectively such that β_1 is a principal type-scheme of some closed BCK lambda-term.

The combinator W ($\equiv \lambda xy.xyy$) corresponds to the contraction rule of Gentzen's LJ. The fact that intuitionistic logic is obtained by adding the contraction rule to BCK logic corresponds to that the combinator S can be constructed with using only the conbinators B, C, K and W. Actually, we can construct S without using K, thus:

$$B(B(BW)B)C =_{c\beta\eta} S.$$

In view of the facts in this note, we had better have named three logics, called LBCA, LBCB and LBCC in [4], LB, LBC and LBK, respectively.

We conclude this note by posing an open problem.

PROBLEM 2.12 (A BCK version of Problem 1.5). Let an implicational formula α be minimal in LBCK. Let M and N be closed BCK lambda-terms in $\beta\eta$-normal form. Then, is it true that M is congruent to N if $\vdash_{TA_\lambda} M \in \alpha$ and $\vdash_{TA_\lambda} N \in \alpha$?

Yes. Hirokawa
Jaskowski + Mints
Tatsuta

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