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A Relation between Strongly Regular Rings and Pseudo-fields

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In our former paper [3], we gave a definition of pseudo-fields and studied them. Prof. P. M. Cohn has pointed out that pseudo-fields are similar to strongly regular rings. Going into details, it is as follows. Usually, a strongly regular ring is defined as a ring which satisfies the following condition: for each element x, there exists an element a such that $ax^2=x$. If we denote such a by x^{-1}, strongly regular rings satisfy our conditions P1–P9 and P11, but don't always satisfy the condition P10. What significance has our condition P10 in the context of strongly regular rings?

The answer to the above question is as follows. If x^* denotes an element a such that $ax^2=x$ and x^{-1} denotes $(x^*)^2x$, then we can show that our condition P10–P12 are satisfied. We show first that P12, (*) $x=x^{-1}x^2$ and (**) $x^{-1}=x(x^{-1})^2$ are satisfied. In the next, we show that P10 and P11 are derivable from P1–P9, (*) and (**).

P12: $\quad 0^{-1}=(0^*)^2 0=0.$

(*): $\quad x^{-1}x^2=(x^*)^2xx^2=x^*(x^*x^2)x=x^*x^2=x.$

(**):
$$\begin{aligned} x(x^{-1})^2&=x(x^*)^2x(x^*)^2x\\ &=(x^*)^4x^3 \qquad \text{(by } x^*x=xx^*)\\ &=(x^*)^3x^*x^2x=(x^*)^3x^2=(x^*)^2x=x^{-1} \end{aligned}$$

The following method deriving P11 is due to Cohn [2].

$$\begin{aligned} xx^{-1}x-x&=(xx^{-1}x-x)^{-1}(xx^{-1}x-x)^2\\ &=(xx^{-1}x-x)^{-1}(xx^{-1}(x^2x^{-1})x-(x^2x^{-1})x-x(x^{-1}x^2)+x^2)\\ &=(xx^{-1}x-x)^{-1}(x(x^{-1}x^2)-x^2-x^2+x^2)\\ &=(xx^{-1}x-x)^{-1}(x^2-x^2-x^2+x^2)\\ &=0. \end{aligned}$$

Hence, $xx^{-1}x=x$. Similarly we prove $x^2x^{-1}=x$, and now $(xy-xyx^{-1}x)=0$; hence we have P11. Similarly we can prove $x^{-1}xy=yx^{-1}x$ and $x^{-1}x=xx^{-1}$.

$$\begin{aligned} &-\{x^{-1}-y^{-1}(1-xx^{-1})\}(x-y)\{y^{-1}-(1-yy^{-1})x^{-1}\}\\ &=-\{(x^{-1}x-x^{-1}y+y^{-1}(1-xx^{-1})y\}\{y^{-1}-(1-yy^{-1})x^{-1}\} \quad \text{(by } x=xx^{-1}x)\\ &=-(x^{-1}x-x^{-1}y+y^{-1}y-y^{-1}yxx^{-1})\{y^{-1}-(1-yy^{-1})x^{-1}\} \end{aligned}$$

$$=-\{x^{-1}xy^{-1}-x^{-1}yy^{-1}+y^{-1}-xx^{-1}y^{-1}-x^{-1}x(1-yy^{-1})x^{-1}\cancel{-y^{-1}x(1-yy^{-1})x^{-1}}\}$$

(by (**), $x^{-1}xy=yx^{-1}x$ and $x^{-1}x=xx^{-1}$)

$$=x^{-1}-y^{-1}.$$

Hence, we have

$$\begin{aligned}
x^{-1}-y^{-1} &= -\{x^{-1}-y^{-1}(1-xx^{-1})\}(x-y)\{y^{-1}-(1-yy^{-1})x^{-1}\} \\
&= -\{x^{-1}-y^{-1}(1-xx^{-1})\}(x-y)(x-y)^{-1}(x-y)\{y^{-1}-(1-yy^{-1})x^{-1}\} \\
&= -(x-y)(x-y)^{-1}\{x^{-1}-y^{-1}(1-xx^{-1})\}(x-y)\{y^{-1}-(1-yy^{-1})x^{-1}\} \\
&= (x-y)(x-y)^{-1}(x^{-1}-y^{-1}).
\end{aligned}$$

Therefore, we can regard any strongly regular ring as a desirable pseudo-field. Theorem 2.8 in [3] and the subdirect reduction theorem in Birkhoff [1] give another proof of Proposition 12.5 in Stenström [4, p. 41].

ERRATA

Free Algebras over All Fields and Pseudo-fields. Rep. Fac. Sci., SHIZUOKA Univ., **9** (1975), 9–15.

Page 13, line 1 frome the bottom. For "pseudo-field", read "pseudo-fields".

Page 15, line 11. For "$A \notin \mathbf{K}$", read "$A \in \mathbf{K}$".

Reference

1) G. Birkhoff, Lattice Theory, *Amer. Math. Soc. Colloq. Publ.* vol. **25**, revised edition, Amer. Math. Soc., New York, N. Y., 1948.

2) P. M. Cohn, private communication, 1975.

3) Y. Komori, Free Algebras over All Fields and Pseudo-fields, *Rep. Fac. Sci., Shizuoka Univ.*, **10** (1975), 9–15.

4) B. Stenstrõm, Rings of Quotients, Springer-Verlag, Berlin Herlin Heidelberg New York, 1975.