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The Separation Theorem of the $\aleph_0$-Valued Lukasiewicz Propositional Logic

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A standard set set of axioms for the $\aleph_0$-valued Lukasiewicz propositional logic is the following.

A1. $p \supset q \supset p$.
A2. $(p \supset q) \supset (q \supset r) \supset p \supset r$.
A3. $p \vee q \supset q \vee p$.
A4. $(p \supset q) \vee (q \supset p)$.
A5. $(\sim p \supset \sim q) \supset q \supset p$.

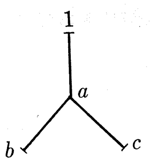
Here we use $P \vee Q$ as the abbreviation of $(P \supset Q) \supset Q$. We assosiate to the right, and use the convention that $\supset$ binds less strongly than the other connectives. The rules of inference are modus ponens, that is, if P and $P \supset Q$, then Q, and the rule of substitution for propositional variables. If a formula contains no connective other than $\supset$, it is called a C formula. A theorem is a formula which is derivable from A1–A5. A C theorem is a C formula which is derivable from A1–A4. Since no primitive logical connective exists other than $\supset$ and $\sim$, the separation theorem is the following.

Separation theorem: *For any C formula P, if P is a theorem, then P is a C theorem.*

Meredith [3] and Chang [2] have shown that A4 is derivable from the rest. But we will show in §1 A4 is not derivable without using A5, that is, the subsystem A1–A3, A5 is not separable. This gives a negative answer to the question by Rose and Rosser (cf. [4] p13). Next, we will show the separation theorem of the full system.

§1. Non separability of the subsystem

Consider the following Hasse diagram. We define the function $\rightarrow$ on $\{1, a, b, c\}$ as follows:



for any x, $1 \to x = x$,
if $x \leqq y$, then $x \to y = 1$,
$a \to b = a \to c = b \to c = c \to b = a$.

1 is the only designated value. We regard this algebra as a model. Then, we can easily shown that if P and $P \supset Q$ are valid, then Q is valid, and that A1–A3 are valid. In the axiom A4, we assign b and c for p and q respectively. We have that the value is a and A4 is not valid. Hence, A4 is not derivable from A1–A3.

§ 2. C algebras

A C algebra is an algebra $<A; 1, \to>$ which satisfies the following axioms, where A is a non empty set and 1 and $\to$ are 0-ary and 2-ary functions on A respectively.

2. 1 $1 \to x = x$.

2. 2 $x \to y \to x = 1$.

2. 3 $(x \to y) \to (y \to z) \to x \to z = 1$.

2. 4 $x \cup y = y \cup x$.

2. 5 $(x \to y) \cup (y \to x) = 1$.

We abbreviate $(x \to y) \to y$ by $x \cup y$. We use the same convention as before. We say simply that A is a C algebra, when $<A; 1, \to>$ is a C algebra. We denote $x \to y = 1$ by $x \leqq y$. Then, we can verify without 2. 5 the following:

2. 6 $x \leqq 1$

2. 7 $x \leqq x$

2. 8 $x \leqq y$ and $y \nleqq z \Rightarrow x \leqq z$

2. 9 $x \leqq y$ and $y \leqq x \Rightarrow x = y$

2. 10 $x \to y \to z = y \to x \to z$

2. 11 $x \leqq x \cup y$ and $y \leqq x \cup y$

2. 12 $x \leqq z$ and $y \leqq z \Rightarrow x \cup y \leqq z$

2. 13 $y \to z \leqq (x \to y) \to x \to z$.

We define the notation $(x\to)^n y$ $(n=0, 1, 2, \cdots)$ as $(x\to)^0 y = y$ and $(x\to)^{n+1} y = x \to (x\to)^n y$. Then, by 2.13 we have

2. 14 $y \to (x\to)^n z \leqq (x\to)^n y \to (x\to)^{n+m} z$.

By 2.6–2.9, the relation $\leqq$ is an order relation with the largest element 1. By 2.11 and 2.12, $x \cup y = \sup(x, y)$. But if we change 2.4 for $x \cup y \to y \cup x = 1$, the relation $\leqq$ becomes no more than a pseudo-order.

Theorem 2. 1. *Let A be a C algebra with a smallest element* 0. *Then, for any elment x, y of A,*

$$(x \to 0) \to y \to 0 \leqq y \to x.$$

Proof By 2.1 and $0 \to y = 1$, when we substitute 0 for x in 2.4, we have (*) $y = (y \to 0) \to 0$. In 2.3, we substitute 0 for z. Then we have $x \to y \leqq (y \to 0) \to x \to 0$. In this, we substitute $x \to 0$ and $y \to 0$ for x and y, respectively. Then by (*) we have $(x \to 0) \to y \to 0 \leqq y \to x$. **Q.E.D.**

The above theorem can be verified without 2.5. By the result of [3] and [2], this means that if an algebra satisfies 2.1–2.4 and has a smallest element, then it satisfies 2.5. Therefore, the algebra in §1 has no smallest element.

Definition 2. 2. Let A be a C algebra. A non-empty subset J of A is *a filter of* A if it satisfies the following two conditions:

1) $1 \in J$
2) $x \in J$ and $x \to y \in J \Rightarrow y \in J$.

Definition 2. 3. Let A be a C algebra and J be a filter of A. We define a relation $\sim_J$ on A as follows:

$$x \sim_J y \Longleftrightarrow x \to y \in J \text{ and } y \to x \ni J.$$

We can easily the following theorems.

Theerem 2. 4. *For any* C *algebra and any filter* J *of* A, *the relation* $\sim_J$ *is a congruence relation and* $A/\sim_J$ *is naturally a* C *algebra.* ($A/\sim_J$ *is denoted by* A/J.)

Theorem 2. 5. (*Homomorphism Theorem*) *Let* A *and* B *be* C *algebras, and* $\varphi: A \to B$ *be a homomorphism of* A *onto* B. *Then* $J = \varphi^{-1}(1_B)$ *is a filter of* A, *and* A/J *is isomorphic to* B, *where the isomorpism is given by* $[a] \mapsto \varphi(a)$ $(a \in A)$. ($[a]$ *is the element of* A/J *which contains* a.)

We now define the term 'irreducible' which is called 'subdirectly irreducible' in Birkhoff [1].

Definition 2. 6. Let A be a C algebra, x be an element of A other than 1. A is *irreducible with respect to* x if x is contained within any filter of A which contains at least an element other than 1. A is *irreducible*, if there exists an element such thet A is irreducible with respect to the element or A has only one element 1.

Lemma 2. 7. *If* J *is a filter generated by* x, *then*

$$J = \{y \mid (x \to)^m y = 1 \text{ for some non-negative integer } m\}.$$

Proof Let K be the right hand side in the above identity. Clearly, $J \supseteq K$. Suppose that $(x \to)^m (u \to v) = 1$ and $(x \to)^n u = 1$. By 2.14, $(x \to)^{n+m} v = (x \to)^n u \to (x \to)^{n+m} v \geqq u \to (x \to)^m v = (x \to)^m (u \to v) = 1$. Hence, $(x \to)^{n+m} v = 1$. Therefore, $K \supseteq J$. **Q.E.D.**

Lemma 2. 8. *For any non-negative integer n,* $(x\to)^n y=1$ and $x\cup y=1 \Rightarrow y=1$.

Proof If $n=0$, clearly $y=1$. Suppose that $(x\to)^{n+1}y=1$ and $x\cup y=1$. Then, $(x\to)^{n+1}y=x\to(x\to)^n y=1$ and $1=x\cup y\leqq x\cup(x\to)^n y=(x\to)^{n+1}y\to(x\to)^n y$. Hence, we have $(x\to)^n y=1$ and $y=1$ by the inductive hypothesis. **Q.E.D.**

By Lemma 2.7 and Lemma 2.8, we have the following lemma.

Lemma 2. 9. *If* $x\cup y=1$ *and* $y\neq 1$, *then the filter generated by x does not contain y.*

Theorem 2. 10. *Any irreducible C algebra is linearly ordered.*

Proof Clearly, the theorem holds when the C algebra has only one element. Lt A be irreducible with respect to z. First, we show that z is comparable with any element. Let x be an element which is not comparable with z. Then, $x\to z\neq 1$ and $z\to x\neq 1$. By 2.5, $(x\to z)\cup(z\to x)=1$. By Lemma 2.9, the filter generated by $z\to x$ does not contain $x\to z$. Hence, this filter does not contain z. This is contradictory to that A is irreducible with respect to z.

Since for any element x such that $x\geqq z$ A is irreducible with respect to x, $A_z=\{x\,|\,x\geqq z\}$ is a linearly ordered set. Hence, if $x\cup y=1$, then $x=1$ or $y=1$ for any $x, y\in A$. Therefore, by 2.5 we have that A is linearly ordered. **Q.E.D.**

In a linearly ordered and finitely genetated C algebra A, if a is the smallest generator, a is the smallest element of A. Hence, we have the following theorem.

Theorem 2. 10. *Any irreducible and finitely generated C algebra is linearly ordered and has a smallest element.*

§ 3. CN algebras

A CN algebra is an algebra $\langle A; 1, \to, \neg\rangle$ which satisfies the following axiom, where $\langle A; 1, \to\rangle$ is a C algebra and $\neg$ is a 1-ary function on A.

3. 1 $\neg x\to\neg y\leqq y\to x$.

We define the relation $\leqq$, filters and etc. similar to C algebras. We can easily verify the following:

3. 2 $\neg 1\to y=1$.

3. 3 $y\to\neg 1=\neg y$.

By 3.2, 3.3 and Theorem 2.1, we have the following theorem.

Theorem 3. 1. *A CN algebra is a C algebra with a smallest element* 0, *and a function* $\neg$ *defined by* $\neg x=x\to 0$. *Conversely, any such C algebra is a CN algebra.*

§ 4. The separation theorem

Let A be a C algebra (or CN algebra) and P be a C formula (or formula). If $f(P)=1$ for any assignment of A, we say that P is valid in A. Let P be a C formula and not a C theorem. Let n be the number of different propositional variables appearing in P. Then, there exists a n-generated C algebra A (for example n-generated Lindenbaum algebra) such that P is not valid in A. Let f be an assingment of A such that $f(P)\neq 1$. Let J be a filter not containing $f(P)$ such that for any filter $K \supsetneqq J$ $f(P)\in K$. By Zorn's lemma, such a filter exists. Then, we can show that A/J is an irreducible and n-generated C algebra and P is not valid in A/J. Since A/J is linearly ordered and has a smallest element, by Theorem 3.1 we can regard A/J as a CN algebra. Generally, if A is a CN algebra and P is not valid in A, then P is not a theorem. Hence, P is not a theorem. Thus the proof is completed.

Recently, the author has known the result by Rose [5] (known only from [6]). His result is essentially equivalent to the separation theorem.

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