YUICHI KOMORI

Syntactical Investigations into BI Logic and $BB'I$ Logic

Abstract. In this note, we will study four implicational logics $B, BI, BB' and BB'I$. In [5], Martin and Meyer proved that a formula α is provable in BB' if and only if α is provable in $BB'I$ and α is not of the form of $\beta \to \beta$. Though it gave a positive solution to the $P - W$ problem, their method was semantical and not easy to grasp. We shall give a syntactical proof of the syntactical relation between BB' and $BB'I$ logics. It also includes a syntactical proof of Powers and Dwyer's theorem that is proved semantically in [5]. Moreover, we shall establish the same relation between B and BI logics as BB' and $BB'I$ logics. This relation seems to say that B logic is meaningful, and so we think that B logic is the weakest among meaningful logics. Therefore, by Theorem 1.1, our Gentzen-type system for BI logic may be regarded as the most basic among all meaningful logics. It should be mentioned here that the first syntactical proof of $P - W$ problem is given by Misao Nagayama [6].

1. Introduction

We will be concerned with four logics of pure implication, formulated in a language constructed in the usual way from a set of propositional variables, with a single binary connective $\to$. We use $p, q, ..., p_1, q_1, ...,$ as variables ranging over propositional variables. We use $\alpha, \beta, ..., \alpha_1, \beta_1, ...,$ as variables ranging over formulas. By the symbol $\equiv$, we mean the equality as sequences of symbols. Parentheses will be omitted following the convention that $\to$ associates to the right. For example, $\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4$ denotes $\alpha_1 \to (\alpha_2 \to (\alpha_3 \to \alpha_4))$.

The first logic, which we call B, has axioms all instances of

$$(B)\ (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma,$$

and a rule

$$\frac{\alpha \to \beta \quad \alpha}{\beta}\ \text{(modus ponens)}.$$

The second logic, BI, has in addition to the axioms and rules of B the axiom scheme

$$(I)\ \alpha \to \alpha.$$

Presented by **Hiroakira Ono**; *Received August* 23, 1993; *Revised November* 11, 1993

Studia Logica **53**: 397–416, 1994.

The third logic, BB', has in addition to the axioms and rules of B the axiom scheme

$$(B')\ (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma.$$

The fourth logic, $BB'I$, has in addition to the axioms and rules of BB' the axiom scheme I.

Let L be any one of logics B, BI, BB' or $BB'I$. We write $L \vdash \alpha$ $(L \nvdash \alpha)$ to mean that α is (resp. is not) provable in the logic L. Such systems as B, BI, BB' and $BB'I$ are called Hilbert-type systems , contrasting with Gentzen-type systems. We say that a formula α is trivial (non-trivial) if α is (resp. is not) of the form $\beta \to \beta$. We have the following three main theorems.

THEOREM 1.1 *For any formula* α, $B \vdash \alpha$ *if and only if* $BI \vdash \alpha$ *and* α *is non-trivial.*

THEOREM 1.2 *For any formula* α, $BB' \vdash \alpha$ *if and only if* $BB'I \vdash \alpha$ *and* α *is non-trivial.*

THEOREM 1.3 *The four logics* B, BI, BB' *and* $BB'I$ *are decidable.*

Anderson and Belnap asked in §**8.11** of [1] whether α and β are the same formula if both formulas $\alpha \to \beta$ and $\beta \to \alpha$ are provable in $BB'I$. This question is known as the $P - W$ problem to which Martin and Meyer [5] has provided a positive solution by a semantical method.

Let us explain why Theorem 1.2 is sufficient for the $P - W$ problem: To argue by contradiction, we assume that the $P - W$ problem did not hold. The assumption implies that there would exist distinct formulas α, β such that $BB'I \vdash \alpha \to \beta$ and $BB'I \vdash \beta \to \alpha$. Since α is distinct from β, by Theorem 1.2 both $\alpha \to \beta$ and $\beta \to \alpha$ are in fact provable in BB'. However then $\beta \to \beta$ would be provable in BB' by the rule of modus ponens and an axiom scheme B, which contradicts Theorem 1.2.

2. The system LBI for the logic BI

Now we shall introduce Gentzen-type systems LB and LBI, where the cut elimination theorem holds for LBI. Using the cut elimination theorem, we will prove that for any formula α, $B \vdash \alpha$ if $BI \vdash \alpha$ and α is non-trivial. This is a B–BI analogue of Powers and Dwyer's theorem which states the similar relation between BB' and $BB'I$.

In the following, we will use Greek capital letters $\Gamma, \Delta, \Sigma ...$ for finite (possibly empty) sequences of formulas separated by commas. We denote the number of formulas in Γ by $|\Gamma|$. An expression of the form $\Gamma \Rightarrow \alpha$ is called a sequent, where Γ is a finite sequence of formulas and α is a formula. Let $\Gamma \equiv \alpha_1, \cdots, \alpha_n$ (sometimes denoted by $(\alpha_1, \cdots, \alpha_n)$) and β be a formula. We denote a formula $\alpha_1 \to \cdots \to \alpha_n \to \beta$ by $\Gamma \to \beta$. We assume here a familiarity with the basic knowledge of Gentzen-type systems (see e.g. [3]).

Initial sequents of *LBI* are of the form

$$p \Rightarrow p \qquad \text{for any propositional variable } p.$$

Rules of inferences of *LBI* are as follows:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \text{(cut)},$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} (\to \text{right}), \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \to \beta, \alpha, \Delta \Rightarrow \gamma} (\to \text{left}).$$

Initial sequents of *LB* are of the form for any formulas α, β and γ and for any sequence Γ,

$$\beta \to \Gamma \to \gamma, \alpha \to \beta, \alpha, \Gamma \Rightarrow \gamma.$$

The rules of *LB* are the same as those of *LBI*.

Next we will introduce an auxiliary Gentzen-type system L^*B to prove that B and *LB* are logically equivalent.

Initial sequents of L^*B are the same as those of *LB*.

Rules of inferences of L^*B are the following:

$$\frac{\Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Delta \Rightarrow \gamma} \text{(cut*)},$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} (\to \text{right}), \qquad \frac{\beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \alpha, \Delta \Rightarrow \gamma} (\to \text{left*}).$$

We write $L \vdash \Gamma \Rightarrow \alpha$ $(L \nvdash \Gamma \Rightarrow \alpha)$ to mean that the sequent $\Gamma \Rightarrow \alpha$ is (resp. is not) provable in the Gentzen-type system L.

LEMMA 2.1 *For any sequence* Γ *of formulas and any formula* α, $B \vdash \Gamma \to \alpha$ *if and only if* $L^*B \vdash \Gamma \Rightarrow \alpha$.

LEMMA 2.2 *For any sequence* Γ *of formulas and any formula* α, $LB \vdash \Gamma \Rightarrow \alpha$ *if and only if* $L^*B \vdash \Gamma \Rightarrow \alpha$.

PROOF. As rules of inferences of L^*B are those of LB and initial sequents of L^*B are the same as those of LB, we have that $LB \vdash \Gamma \Rightarrow \alpha$ if $L^*B \vdash \Gamma \Rightarrow \alpha$. So, it suffices to show that both rules (cut) and ($\rightarrow$ left) are derived rules of L^*B. The following proof figure shows that (cut) is derived:

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{\Rightarrow \Gamma \rightarrow \alpha}\,(\rightarrow \text{right}) \qquad \dfrac{\alpha, \Delta \Rightarrow \gamma}{\Gamma \rightarrow \alpha, \Gamma, \Delta \Rightarrow \gamma}\,(\rightarrow \text{left}^*)}{\Gamma, \Delta \Rightarrow \gamma}\,(\text{cut}^*) \quad .$$

The following proof figure shows that ($\rightarrow$ left) is derived:

$$\dfrac{\dfrac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma \Rightarrow \beta \rightarrow \Delta \rightarrow \gamma}\,(\rightarrow \text{right}) \qquad \overset{\text{initial sequent}}{\beta \rightarrow \Delta \rightarrow \gamma, \alpha \rightarrow \beta, \alpha, \Delta \Rightarrow \gamma}}{\Gamma, \alpha \rightarrow \beta, \alpha, \Delta \Rightarrow \gamma}\,(\text{cut}) .$$

■

From Lemma 2.1 and Lemma 2.2, the next theorem, which tells that the system LB is logically equivalent to B logic, follows immediately.

THEOREM 2.3 *For any sequence* Γ *of formulas and any formula* α, $B \vdash \Gamma \rightarrow \alpha$ *if and only if* $LB \vdash \Gamma \Rightarrow \alpha$.

In the same way as the above, we can prove the following.

THEOREM 2.4 *For and any sequence* Γ *of formulas and any formula* α, $BI \vdash \Gamma \rightarrow \alpha$ *if and only if* $LBI \vdash \Gamma \Rightarrow \alpha$.

Next we will prove the cut elimination theorem for LBI. Our theorem can be proved in the standard way (see [3]). But the fact that LBI does not have the contraction rule will simplify the proof at two points. The first point is that we need not replace each cut rule by a mix rule. The second point is that we can prove it by single induction on the length, not double induction on the degree and the rank. The *length* of the proof P, denoted by $L(P)$, is the number of sequents appearing in P. We denote the endsequent of a proof P by $end(P)$. Our theorem makes mention also of the length of the cut-free proof.

THEOREM 2.5 CUT ELIMINATION THEOREM FOR LBI *If* P *is a proof of* LBI, *then there exists a cut-free proof* Q *of* LBI *such that* $L(Q) \leqslant L(P)$ *and* $end(Q) \equiv end(P)$.

PROOF. It suffices to show that

(*) *If P is a proof of LBI, which contains only one cut rule, occurring as the last inference, then there exits a cut-free proof Q such that $L(Q) < L(P)$ and $end(Q) \equiv end(P)$.*

So, let us suppose that P is a proof which contains a cut only as the last inference:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ (\text{cut}).$$

We prove (*) by induction on the length of P. We divide the proof into the following three cases:

Case 1. Either $\Gamma \Rightarrow \alpha$ or $\alpha, \Delta \Rightarrow \gamma$ is an initial sequent.

Case 2. Either $\Gamma \Rightarrow \alpha$ or $\alpha, \Delta \Rightarrow \gamma$ is a lower sequent of a rule, whose principal formula is not (the occurence of) α, to which the cut rule is applied.

Case 3. Both $\Gamma \Rightarrow \alpha$ and $\alpha, \Delta \Rightarrow \gamma$ are lower sequents of some rules such that principal formulas of both rules are (occurrences of) α to which the cut rule is applied.

In the following, we will show (*) only for Case 3 where α is of the form $\beta \to \delta$. In this case, the last part of P is of the form

$$\frac{\dfrac{\Gamma, \beta \Rightarrow \delta}{\Gamma \Rightarrow \beta \to \delta} \quad \dfrac{\delta, \Delta \Rightarrow \gamma}{\beta \to \delta, \beta, \Delta \Rightarrow \gamma}}{\Gamma, \beta, \Delta \Rightarrow \gamma}\ (\text{cut}).$$

Consider the proof P' of which the last part is of the form

$$\frac{\Gamma, \beta \Rightarrow \delta \quad \delta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \Delta \Rightarrow \gamma}\ (\text{cut}).$$

Because $L(P') < L(P)$, we can eliminate this cut rule by the induction hypothesis. ■

In our joint paper [7] with Ono, we proved the cut elimination theorem for logics without the contraction rule by double induction on the grade and the length. However we could prove it by induction on the length like the above.

R. Hindley asked the author whether the implicational part of $LBCA$ logic is logically equivalent to BI logic, where $LBCA$ (badly named in [4]) is obtained from the intutionistic logic LJ by deleting all structure rules. We can answer the question in the negative here. Whereas $(p \to p) \to q \Rightarrow q$ is provable in $LBCA$, by the cut elimination theorem in the above we have that it is not provable in LBI. So, the implicational part of $LBCA$ logic is not logically equivalent to BI logic. All rules of LBI are derivable in $LBCA$, and so the logic BI is weaker than $LBCA$.

Similarly to the proof of Theorem 6.6 of [10], we can derive the following theorem from Theorem 2.5.

THEOREM 2.6 *Craig's interpolation theorem holds for LBI. More precisely, if $LBI \vdash \Gamma, \Delta \Rightarrow \gamma$ and $\Gamma \neq \emptyset$ then there exists a formula α such that*

1. *$LBI \vdash \Gamma \Rightarrow \alpha$ and $LBI \vdash \alpha, \Delta \Rightarrow \gamma$, and*

2. *all propositional variables in α occur both in Γ and $\Delta \cup \{\gamma\}$.*

By the cut elimination theorem and lack of the contraction rule, only finite number of sequents can occur in a cut-free proof of $\Gamma \Rightarrow \gamma$ for any sequent $\Gamma \Rightarrow \gamma$. So, by Theorem 2.4, we have

THEOREM 2.7 *The logic BI is decidable.*

We are at the point of showing the same relation between B and BI logics as what Powers and Dwyer showed between BB' and $BB'I$ logics.

A sequent $\Gamma \Rightarrow \gamma$ is called trivial (non-trivial) if the formula $\Gamma \rightarrow \gamma$ is trivial (resp. non-trivial).

THEOREM 2.8 *For any sequent $\Gamma \Rightarrow \gamma$, $LB \vdash \Gamma \Rightarrow \gamma$ if $LBI \vdash \Gamma \Rightarrow \gamma$ and $\Gamma \Rightarrow \gamma$ is non-trivial.*

PROOF. We prove this by induction on the length of the cut-free LBI proof of the sequent $\Gamma \Rightarrow \gamma$. We divide the proof into the following two cases:

Case 1. $\Gamma \Rightarrow \sim$ is a lower sequent of ($\rightarrow$ right).

Case 2. $\Gamma \Rightarrow \gamma$ is a lower sequent of ($\rightarrow$ left).

For Case 1, it is trivial. In Case 2, the last part of the proof is of the form

$$\frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta, \alpha, \Delta \Rightarrow \gamma} (\rightarrow \text{left}).$$

If $\Gamma, \beta, \Delta \Rightarrow \gamma$ is non-trivial, then we have $LB \vdash \Gamma, \beta, \Delta \Rightarrow \gamma$ by the induction hypothesis. So, we have $LB \vdash \Gamma, (\alpha \rightarrow \beta), \alpha, \Delta \Rightarrow \gamma$.

If $\Gamma, \beta, \Delta \Rightarrow \gamma$ is trivial then Γ is not empty, since otherwise $\Gamma, \alpha \rightarrow \beta, \alpha, \Delta \Rightarrow \gamma$ becomes also trivial. Let $\Gamma \equiv \delta, \Sigma$, then $\delta \equiv \Sigma \rightarrow \beta \rightarrow \Delta \rightarrow \gamma$. So, the lower sequent equals $\Sigma \rightarrow \beta \rightarrow \Delta \rightarrow \gamma, \Sigma, \alpha \rightarrow \beta, \alpha, \Delta \Rightarrow \gamma$. The following proof figure shows that the lower sequent is provable in LB:

$$\dfrac{\dfrac{\text{initial sequent}}{\beta \to \Delta \to \gamma, \alpha \to \beta, \alpha, \Delta \Rightarrow \gamma}}{\Sigma \to \beta \to \Delta \to \gamma, \Sigma, \alpha \to \beta, \alpha, \Delta \Rightarrow \gamma}\ (\to \text{left}).$$

■

To show the converse of Theorem 2.8, it is sufficient to prove that any trivial sequent is not provable in LB. In fact, this will be proved in Theorem 4.3 in a stronger form.

3. The system $LBB'I$ for the logic $BB'I$

Now we shall introduce Gentzen-type systems LBB' and $LBB'I$, for both of which the cut elimination theorem holds. Using the cut elimination theorem, we will prove syntactically Powers and Dwyer's theorem.

To explain the system $LBB'I$, we need an operation called a *guarded merge* on sequences of formulas, originally introduced in [5]. Here by a merge of sequences Γ and Δ, we mean a new sequence exactly consisting of members of Γ and Δ as multisets, in which both Γ and Δ preserve their original orders.

DEFINITION 3.1 A *guarded merge* of Γ and Δ, denoted by $\Gamma \circ \Delta$, is a sequence obtained from a merge of sequences Γ and Δ such that its rightmost formula is that of Δ if $\Delta \neq \emptyset$. $\Gamma \circ \Delta \equiv \Gamma$ if $\Delta = \emptyset$.

Parentheses will be omitted following the convention that $\circ$ associates to the right.

Initial sequents of $LBB'I$ are of the form

$$p \Rightarrow p \qquad \text{for any propositional variable } p.$$

Rules of inferences of $LBB'I$ are the following three.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta \circ \Gamma, \Sigma \Rightarrow \gamma}\ (cut),$$

where $\Gamma \neq \emptyset$ or $\Delta = \emptyset$.

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to \text{right}).$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta \circ (\alpha \to \beta) \circ \Gamma, \Sigma \Rightarrow \gamma}\ (\to \text{left}),$$

where $\Gamma \neq \emptyset$.

Initial sequents of LBB' are of the form, for any propositional variable p, any formulas α and β and any sequence Γ,

$$(\beta \to \Gamma \to p) \circ (\alpha \to \beta, \alpha), \Gamma \Rightarrow p.$$

The system LBB' has five rules of inferences. Three of them are the same as those of $LBB'I$. Additional two rules are as follows.

$$\frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma \circ (\alpha \to \beta, \alpha), \Delta \Rightarrow \gamma} \ (\to \text{left } 1),$$

where we call (the occurrence of) the formula α a *semi-principal* formula of ($\to$ left 1).

$$\frac{\Gamma \Rightarrow \alpha}{(\alpha \to \Delta \to p) \circ \Gamma, \Delta \Rightarrow p} \ (\to \text{left } 2),$$

where $\Gamma \neq \emptyset$ and p is a propositional variable, and we say that (occurrences of) formulas in Δ and the propositional variable p are *semi-principal* formulas of ($\to$ left 2).

Next we will introduce an auxiliary Gentzen-type system L^*BB' to prove that BB' and LBB' are logically equivalent. Initial sequents of L^*BB' are the same as those of LBB'.

Rules of inferences of L^*BB' are the following:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \ (\text{cut}^*),$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\to \text{right}), \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \to \beta, \alpha, \Delta \Rightarrow \gamma} \ (\to \text{left}^*).$$

Clearly, $L^*BB' \vdash \Gamma \Rightarrow \gamma$ if $BB' \vdash \Gamma \to \gamma$. Conversely, $BB' \vdash \Gamma \to \gamma$ if $L^*BB' \vdash \Gamma \Rightarrow \gamma$ because all rules of L^*BB' are those of LB. So we have

LEMMA 3.2 *For any sequence Γ of formulas and any formula α, $BB' \vdash \Gamma \to \alpha$ if and only if $L^*BB' \vdash \Gamma \Rightarrow \alpha$.*

We need the following three lemmas to prove Lemma 3.6 corresponding to Lemma 2.2.

LEMMA 3.3 *The following rule (cut**) is derived in L^*BB':*

$$\frac{\beta \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta, \beta, \Sigma \Rightarrow \gamma} \ (cut**).$$

PROOF. The following proof figure shows that (cut**) is derived:

$$\dfrac{\dfrac{\Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta \Rightarrow \alpha \to \Sigma \to \gamma}\,(\to \mathrm{r}) \qquad \dfrac{\dfrac{\beta \Rightarrow \alpha}{\Rightarrow \beta \to \alpha}\,(\to \mathrm{r}) \quad \dfrac{\text{initial sequent}}{\beta \to \alpha, \alpha \to \Sigma \to \gamma, \beta, \Sigma \Rightarrow \gamma}\,(\to \mathrm{r})}{\alpha \to \Sigma \to \gamma, \beta, \Sigma \Rightarrow \gamma}\,(\mathrm{c}^*)}{\Delta, \beta, \Sigma \Rightarrow \gamma}\,(\mathrm{c}^*)$$

■

LEMMA 3.4 *The following rule ($\to$ left**) is derived in L^*BB':*

$$\frac{\delta, \beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \delta, \alpha, \Delta \Rightarrow \gamma}\,(\to \mathit{left}^{**}).$$

PROOF. The following proof figure shows that ($\to$ left**) is derived:

$$\dfrac{\dfrac{\delta, \beta, \Delta \Rightarrow \gamma}{\delta \Rightarrow \beta \to \Delta \to \gamma}\,(\to \text{right}) \qquad \dfrac{\text{initial sequent}}{\alpha \to \beta, \beta \to \Delta \to \gamma, \alpha, \Delta \Rightarrow \gamma}\,(\to \text{right})}{\alpha \to \beta, \delta, \alpha, \Delta \Rightarrow \gamma}\,(\text{cut}^{**})$$

■

LEMMA 3.5 *The following rule ($\to$ left***) is derived in L^*BB':*

$$\frac{\Gamma, \Delta, \beta, \Sigma \Rightarrow \gamma}{\Gamma, \alpha \to \beta, \Delta, \alpha, \Sigma \Rightarrow \gamma}\,(\to \mathit{left}^{***}).$$

PROOF. When $\Delta = \emptyset$, ($\to$ left***) equals ($\to$ left*). Let $\Delta \equiv \delta, \Theta$. The following proof figure shows that ($\to$ left***) is derived:

$$\dfrac{\dfrac{\Gamma, \Delta, \beta, \Sigma \Rightarrow \gamma}{\Gamma \Rightarrow \Delta \to \beta \to \Sigma \to \gamma}\,(\to \mathrm{r}) \qquad \dfrac{\dfrac{\dfrac{\text{initial sequent}}{\alpha \to \beta, \beta \to \Sigma \to \gamma, \alpha, \Sigma \Rightarrow \gamma}\,(\to \mathrm{r})}{\alpha \to \beta, \Theta \to \beta \to \Sigma \to \gamma, \Theta, \alpha, \Sigma \Rightarrow \gamma}\,(\to \mathrm{l}^*)}{\Delta \to \beta \to \Sigma \to \gamma, \alpha \to \beta, \Delta, \alpha, \Sigma \Rightarrow \gamma}\,\to (\mathrm{l}^{**})}{\Gamma, \alpha \to \beta, \Delta, \alpha, \Sigma \Rightarrow \gamma}\,(\mathrm{c}^*)$$

■

LEMMA 3.6 *For any sequence* Γ *of formulas and any formula* α, $LBB' \vdash \Gamma \Rightarrow \alpha$ *if and only if* $L^*BB' \vdash \Gamma \Rightarrow \alpha$.

PROOF. As all rules of inferences of L^*BB' are those of LBB' and initial sequents of L^*BB' are the same as those of LBB', $LBB' \vdash \Gamma \Rightarrow \alpha$ if $L^*BB' \vdash \Gamma \Rightarrow \alpha$. The rule ($\rightarrow$ left 1) is equal to ($\rightarrow$ left***). So, it suffices to show that the rules (cut), ($\rightarrow$ left) and ($\rightarrow$ left 2) are derived rules of L^*BB'.

We prove by induction on $|\Gamma|$ that (cut) is derived. When $|\Gamma| = 1$, (cut) equals (cut**). Let $\Gamma \equiv \Theta, \delta$. The following proof figure shows that (cut) is derived in this case:

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{\Theta \Rightarrow \delta \rightarrow \alpha}(\rightarrow \text{right}) \quad \dfrac{\Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta \circ (\delta \rightarrow \alpha, \delta), \Sigma \Rightarrow \gamma}(\rightarrow \text{left***})}{\Delta \circ (\Theta, \delta), \Sigma \Rightarrow \gamma}(\text{the induction hypothesis}).$$

The following proof figure shows that ($\rightarrow$ left) is derived:

$$\dfrac{\Gamma \Rightarrow \alpha \quad \dfrac{\Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta \circ (\alpha \rightarrow \beta, \alpha), \Sigma \Rightarrow \gamma}(\rightarrow \text{left***})}{\Delta \circ (\alpha \rightarrow \beta) \circ \Gamma, \Sigma \Rightarrow \gamma}(\text{cut}) .$$

The following proof figure shows that ($\rightarrow$ left 2) is derived:

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{\Sigma \Rightarrow \beta \rightarrow \alpha}(\rightarrow \text{right}) \quad \begin{array}{c}\text{initial sequent}\\ (\alpha \rightarrow \Delta \rightarrow p) \circ (\beta \rightarrow \alpha, \beta), \Delta \Rightarrow p\end{array}}{(\alpha \rightarrow \Delta \rightarrow p) \circ \Gamma, \Delta \Rightarrow p}(\text{cut}),$$

where $\Gamma \equiv \Sigma, \beta$. ■

From Lemma 3.2 and Lemma 3.6, the next theorem follows immediately.

THEOREM 3.7 *For any sequence* Γ *of formulas and any formula* α, $BB' \vdash \Gamma \rightarrow \alpha$ *if and only if* $LBB' \vdash \Gamma \Rightarrow \alpha$.

In the same way as the above, we can prove the following.

THEOREM 3.8 *For any sequence* Γ *of formulas and any formula* α, $BB'I \vdash \Gamma \rightarrow \alpha$ *if and only if* $LBB'I \vdash \Gamma \Rightarrow \alpha$.

In the system $LBB'I$, the *length* of the proof P, denoted by $L(P)$, is defined as same as in the system LBI. But in the system LBB', we have to give the following definition, though it is essentially same as that in $LBB'I$. In the below, $|\alpha|$ denotes the number of (occurrences of) the logical connective $\to$ appearing in a formula α. For examples, $|p| = 0$ and $|p \to q| = 1$.

DEFINITION 3.9 Let P be a proof of LBB'. The *length* $L(P)$ of P is defined recursively as follows.

If $\Gamma \Rightarrow p$ is initial sequent, then

$L(\Gamma \Rightarrow p) =$ the number of the logical connective $\to$ appearing in Γ.

If (#) be (cut) or ($\to$ left), then

$$L\left(\frac{P \quad Q}{\Gamma \Rightarrow \gamma}\ (\#)\right) = L(P) + L(Q) + 1.$$

$$L\left(\frac{P}{\Gamma \Rightarrow \alpha \to \beta}\ (\to \text{right})\right) = L(P) + 1.$$

$$L\left(\frac{P}{\Gamma \circ (\alpha \to \beta, \alpha), \Delta \Rightarrow \gamma}\ (\to \text{left } 1)\right) = L(P) + 2|\alpha| + 1.$$

$$L\left(\frac{P}{(\alpha \to \Delta \to p) \circ \Gamma, \Delta \Rightarrow p}\ (\to \text{left } 2)\right) = L(P) + 2|\Delta \to p| - |\Delta| + 1.$$

By the above definition, if a proof P contains no cut, then $L(P)$ is equal to the number of (occurrences of) the logical connective $\to$ appearing in $end(P)$. We need the following lemma to prove the cut elimination theorem for LBB'.

LEMMA 3.10 *For any sequence Γ of formulas and any formulas α and β, if a sequent $\Gamma \Rightarrow \alpha \to \beta$ has a proof P of LBB', then $\Gamma, \alpha \Rightarrow \beta$ has a proof Q of LBB' such that $L(Q) < L(P)$.*

PROOF. By induction on the length of P. ■

Next we will prove the cut elimination theorem for LBB'. The theorem can be proved in the same way as the system LBI except the case of initial sequents and the case of semi-principal formulas.

THEOREM 3.11 CUT ELIMINATION THEOREM FOR LBB' *If P is a proof of LBB', then there exists a cut-free proof Q of LBB' such that $L(Q) \leqslant L(P)$ and $end(Q) \equiv end(P)$.*

PROOF. It suffices to show that

(*) *If P is a proof of LBB', which contains only one cut rule, occurring as the last inference, then there exits a cut-free proof Q of LBB' such that $L(Q) < L(P)$ and $end(Q) \equiv end(P)$.*

So, let us suppose that P is a proof which contains a cut only as the last inference:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta \circ \Gamma, \Sigma \Rightarrow \gamma} \text{(cut)}.$$

We prove (*) by induction on the length of P. In the following, principal formulas are defined as usual and semi-principal formulas are defined in the definitions of the rules ($\rightarrow$ left 1) and ($\rightarrow$ left 2). We divide the proof into the following four cases:

Case 1. $\Gamma \Rightarrow \alpha$ is an initial sequent.

Case 2. $\Delta, \alpha, \Sigma \Rightarrow \gamma$ is an initial sequent.

Case 3. Either $\Gamma \Rightarrow \alpha$ or $\Delta, \alpha, \Sigma \Rightarrow \gamma$ is a lower sequent of a rule such that (the occurence of) α is neither a principal formula nor a semi-principal formula of the rule.

Case 4. Both $\Gamma \Rightarrow \alpha$ and $\Delta, \alpha, \Sigma \Rightarrow \gamma$ are lower sequents of some rules such that principal formulas of both rules are (occurrences of) α.

Case 5. $\Delta, \alpha, \Sigma \Rightarrow \gamma$ is a lower sequent of ($\rightarrow$ left 1) or ($\rightarrow$ left 2) and (the occurence of) α is a semi-principal formula of the rule.

Case 6. $\Gamma \Rightarrow \alpha$ is a lower sequent of ($\rightarrow$ left 2) and (the occurence of) α is a semi-principal formula of the rule.

In the following, we will show (*) only for Case 2, Case 4 and Case 6.

In Case 2, the last part of P is of the form

$$\frac{\Gamma \Rightarrow \alpha \quad (\beta \rightarrow \Delta \rightarrow p) \circ (\gamma \rightarrow \beta, \gamma), \Delta \Rightarrow p}{\Sigma \Rightarrow p} \text{(cut)}.$$

We divide Case 2 into the following four subcases:

(Subcase 2.1) $\alpha \equiv \beta \rightarrow \Delta \rightarrow p$,

(Subcase 2.2) $\alpha \equiv \gamma \rightarrow \beta$,

(Subcase 2.3) $\alpha \equiv \gamma$,

(Subcase 2.4) $\alpha \in \Delta$.

We will show here only for Subcase 2.1 and Subcase 2.4.

In Subcase 2.1, the last part of P is of the form

$$\frac{\Gamma \Rightarrow \beta \to \Delta \to p \quad (\beta \to \Delta \to p) \circ (\gamma \to \beta, \gamma), \Delta \Rightarrow p}{\Gamma \circ (\gamma \to \beta, \gamma), \Delta \Rightarrow p} \text{(cut)}.$$

Consider the proof P' of which the last part is of the form

$$\frac{\Gamma, \beta, \Delta \Rightarrow p}{\Gamma \circ (\gamma \to \beta, \gamma), \Delta \Rightarrow p} (\to \text{left } 1).$$

By Lemma 3.10, the sequent $\Gamma, \beta, \Delta \Rightarrow p$ has a cut-free proof whose length is less than $L(P)-1$. So, we have a cut-free proof Q of $\Gamma \circ (\gamma \to \beta, \gamma), \Delta \Rightarrow p$ such that $L(Q) < L(P)$.

In Subcase 2.4, the last part of P is of the form

$$\frac{\Gamma \Rightarrow \alpha \quad (\beta \to \Delta \to p) \circ (\gamma \to \beta, \gamma), \Sigma, \alpha, \Theta \Rightarrow p}{((\beta \to \Delta \to p) \circ (\gamma \to \beta, \gamma), \Sigma) \circ \Gamma, \Theta \Rightarrow p} \text{(cut)},$$

where $\Delta \equiv \Sigma, \alpha, \Theta$. Consider the cut-free proof Q of which the last part is of the form

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \Rightarrow \alpha}{(\alpha \to \Theta \to p) \circ \Gamma, \Theta \Rightarrow p} (\to \text{left } 2)}{(\Delta \to p, \Sigma) \circ \Gamma, \Theta \Rightarrow p} (\to \text{left } 1)}{(\beta \to \Delta \to p, \beta, \Sigma) \circ \Gamma, \Theta \Rightarrow p} (\to \text{left } 1)}{((\beta \to \Delta \to p) \circ (\gamma \to \beta, \gamma), \Sigma) \circ \Gamma, \Theta \Rightarrow p} (\to \text{left } 1).$$

We can easily prove that $L(Q) < L(P)$.

In Case 4, α is of the form $\beta \to \delta$. We divide Case 4 into the following three subcases:

(Subcase 4.1) α is a principal formula of ($\to$ left),

(Subcase 4.2) α is a principal formula of ($\to$ left 1),

(Subcase 4.3) α is a principal formula of ($\to$ left 2).

We will show here only for Subcase 4.1.

In Subcase 4.1, the last part of P is of the form

$$\frac{\cfrac{\Gamma, \beta \Rightarrow \delta}{\Gamma \Rightarrow \beta \to \delta} \quad \cfrac{\Delta \Rightarrow \beta \quad \Sigma, \delta, \Theta \Rightarrow \gamma}{\Sigma \circ (\beta \to \delta) \circ \Delta, \Theta \Rightarrow \gamma}}{\Sigma \circ \Gamma \circ \Delta, \Theta \Rightarrow \gamma} \text{(cut)}.$$

Consider the proof P' of which the last part is of the form

$$\dfrac{\dfrac{\Delta \Rightarrow \beta \quad \Gamma, \beta \Rightarrow \delta}{\Gamma \circ \Delta \Rightarrow \delta}\text{(cut)–1} \qquad \Sigma, \delta, \Theta \Rightarrow \gamma}{\Sigma \circ \Gamma \circ \Delta, \Theta \Rightarrow \gamma}\text{(cut)–2}.$$

We can eliminate (cut)–1 by the induction hypothesis. The length of the proof obtained from P' eliminating (cut)–1 is less than the length of P. So, we can also eliminate (cut)–2 by the induction hypothesis and the length of the obtained proof is less than the length of P.

In Case 6, the last part of P is of the form

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{(\alpha \to \Theta \to p) \circ \Gamma, \Theta \Rightarrow p}(\to \text{left } 2) \qquad \Delta, p, \Sigma \Rightarrow \gamma}{\Delta \circ ((\alpha \to \Theta \to p) \circ \Gamma, \Theta), \Sigma \Rightarrow \gamma}\text{(cut)}.$$

Consider the cut-free proof Q whose last part is of the form

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \dfrac{\Delta, p, \Sigma \Rightarrow \gamma}{\Delta \circ (\Theta \to p, \Theta), \Sigma \Rightarrow \gamma}(\to \text{left } 1)}{\Delta \circ ((\alpha \to \Theta \to p) \circ \Gamma, \Theta), \Sigma \Rightarrow \gamma}(\to \text{left}).$$

We can easily prove that $L(Q) < L(P)$. ■

The cut elimination theorem for $LBB'I$ can be proved in an easier way than that for LBB'.

THEOREM 3.12 CUT ELIMINATION THEOREM FOR $LBB'I$ *If P is a proof of $LBB'I$, then there exists a cut-free proof Q of $LBB'I$ such that $L(Q) \leqslant L(P)$ and $end(Q) \equiv end(P)$.*

Recently, the author has known that the system $LBB'I$ is the same as the Gentzen formulation for the system $\mathbf{T}_{\to} - \mathbf{W}$ in §**8.11** of [1]. But the cut elimination theorem for the formulation has not been proved in [1].

By the cut elimination theorem and lack of the contraction rule, only finite number of sequents can occur in a cut-free proof of $\Gamma \Rightarrow \gamma$ for any sequent $\Gamma \Rightarrow \gamma$. So, by Theorem 3.7 and Theorem 3.8, we have

THEOREM 3.13 *Both logics BB' and $BB'I$ are decidable.*

Now, we shall prove Powers and Dwyer's theorem.

THEOREM 3.14 POWERS AND DWYER'S THEOREM *For any sequent $\Gamma \Rightarrow \gamma$, $LBB' \vdash \Gamma \Rightarrow \gamma$ if $LBB'I \vdash \Gamma \Rightarrow \gamma$ and $\Gamma \Rightarrow \gamma$ is non-trivial.*

PROOF. We prove this by induction on the length of the cut-free $LBB'I$ proof of the sequent $\Gamma \Rightarrow \gamma$. We divide the proof into the following two cases:

Case 1. $\Gamma \Rightarrow \gamma$ is a lower sequent of ($\rightarrow$ right).

Case 2. $\Gamma \Rightarrow \gamma$ is a lower sequent of ($\rightarrow$ left).

For Case 1, it is trivial. In Case 2, the last part of the proof is of the form

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta \circ (\alpha \rightarrow \beta) \circ \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow \text{left}).$$

We divide Case 2 into the following four subcases:

Subcase 2.1. Both $\Gamma \Rightarrow \alpha$ and $\Delta, \beta, \Sigma \Rightarrow \gamma$ are non-trivial.

Subcase 2.2. $\Gamma \Rightarrow \alpha$ is non-trivial and $\Delta, \beta, \Sigma \Rightarrow \gamma$ is trivial.

Subcase 2.3. $\Gamma \Rightarrow \alpha$ is trivial and $\Delta, \beta, \Sigma \Rightarrow \gamma$ is non-trivial.

Subcase 2.4. Both $\Gamma \Rightarrow \alpha$ and $\Delta, \beta, \Sigma \Rightarrow \gamma$ are trivial.

In the following, we will show it only for Subcase 2.2. We divide Subcase 2.2 into the following two cases: (2.2.1) $\Delta = \emptyset$, (2.2.2) $\Delta \equiv \delta, \Theta$.

(2.2.1) The endsequent is $(\alpha \rightarrow \Sigma \rightarrow \Pi \rightarrow p) \circ \Gamma, \Sigma \Rightarrow \gamma$, where $\gamma \equiv \Pi \rightarrow p$ and $\beta \equiv \Sigma \rightarrow \Pi \rightarrow p$. Consider the proof of which the last part is of the form

$$\frac{\dfrac{\Gamma \Rightarrow \alpha}{(\alpha \rightarrow \Sigma \rightarrow \Pi \rightarrow p) \circ \Gamma, \Sigma, \Pi \Rightarrow p} (\rightarrow \text{left } 2)}{(\alpha \rightarrow \Sigma \rightarrow \Pi \rightarrow p) \circ \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow \text{right}) .$$

By the induction hypothesis, the sequent $\Gamma \Rightarrow \alpha$ is provable in LBB'. So, this shows that the endsequent is provable in LBB'.

(2.2.2) The endsequent is $(\Theta \rightarrow \beta \rightarrow \Sigma \rightarrow \Pi \rightarrow p, \Theta) \circ (\alpha \rightarrow \Lambda \rightarrow q) \circ \Gamma, \Sigma \Rightarrow \gamma$, where $\gamma \equiv \Pi \rightarrow p$, $\beta \equiv \Lambda \rightarrow q$ and $\delta \equiv \Theta \rightarrow \beta \rightarrow \Sigma \rightarrow \Pi \rightarrow p$. Consider the proof of which the last part is of the form

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \alpha}{(\alpha \rightarrow \beta) \circ \Gamma, \Lambda \Rightarrow q} (\rightarrow \text{left } 2)}{(\alpha \rightarrow \beta) \circ \Gamma \Rightarrow \beta} (\rightarrow \text{right})}{(\beta \rightarrow \Sigma \rightarrow \gamma) \circ (\alpha \rightarrow \beta) \circ \Gamma, \Sigma, \Pi \Rightarrow p} (\rightarrow \text{left } 2)}{(\beta \rightarrow \Sigma \rightarrow \Pi \rightarrow p) \circ (\alpha \rightarrow \Lambda \rightarrow q) \circ \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow \text{right})}{(\Theta \rightarrow \beta \rightarrow \Sigma \rightarrow \Pi \rightarrow p, \Theta) \circ (\alpha \rightarrow \Lambda \rightarrow q) \circ \Gamma, \Sigma \Rightarrow \gamma} (\rightarrow \text{left } 1).$$

By the induction hypothesis, the sequent $\Gamma \Rightarrow \alpha$ is provable in LBB'. So, this shows that the endsequent is provable in LBB'. ■

4. Proof of Main Theorem

THEOREM 4.1 *For any propositional variable p, the formula $p \to p$ is not provable in BB'.*

PROOF. By the cut elimination theorem for the system LBB'. ■

Henceforth, a formula α is only used to denote a formula $\alpha_1 \to \cdots \alpha_n \to p$. Hence the number n associated to α may be treated implicitly. Under such a circumstance, Nagayama [6] introduced the following notation:

Notation. We denote a formula $\alpha_i \to \cdots \alpha_n \to p$ by $\overline{\alpha_i}$. In particular, we can write α as $\overline{\alpha_1}$.

Nagayama showed in [6] the following normalization theorem for the system $LBB'I$ using the cut elimination theorem for $LBB'I$.

THEOREM 4.2 NORMALIZATION THEOREM FOR $LBB'I$ *Assume $\Delta, \alpha, \Gamma \Rightarrow \gamma$ is provable in $LBB'I$. Then there are $\Gamma_i, \Sigma_i, \Delta_i$ $(0 \leqslant i \leqslant n)$ and a cut-free proof of $\Delta, \alpha, \Gamma \Rightarrow \gamma$ in $LBB'I$ such that:*

$$\begin{array}{c}
\vdots \qquad\qquad \vdots \\
\Sigma_n \Rightarrow \alpha_n \quad \Delta_n, p, \Gamma_n \Rightarrow p \\
\vdots \\
\dfrac{\begin{array}{c}\vdots \\ \Sigma_1 \Rightarrow \alpha_1\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Sigma_i \Rightarrow \alpha_i\end{array} \qquad \begin{array}{c}\vdots\\ \Delta_i, \overline{\alpha_{i+1}}, \Gamma_i \Rightarrow p\end{array}}{\Delta_{i-1}, \overline{\alpha_i}, \Gamma_{i-1} \Rightarrow p}\ (\to \mathit{left}) \\ \vdots \\ \Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p \end{array}}{\Delta_0, \alpha, \Gamma_0 \Rightarrow p}\ (\to \mathit{left}) \\
\vdots \\
\Delta, \alpha, \Gamma \Rightarrow \gamma,
\end{array}$$

where
(1) a part of the proof, which we call a tail*:*

$$\begin{array}{c}\Delta_0, \alpha, \Gamma_0 \Rightarrow p \\ \vdots \\ \Delta, \alpha, \Gamma \Rightarrow \gamma\end{array}$$

does not contain any inference rule whose principal formula is (the occurence of) α.
(2) $\Delta_{i-1}, \overline{\alpha_i}, \Gamma_{i-1} \equiv \Delta_i \circ (\overline{\alpha_{i+1}}) \circ \Sigma_i, \Gamma_i$ $(1 \leqslant i \leqslant n)$.

We call the proof above, a *normalized proof* w.r.t. α of $\Delta, \alpha, \Gamma \Rightarrow \gamma$. Notice here that by the requirement for the ($\rightarrow$ left) rule of $LBB'I$, every Σ_i in the above proof must be nonempty. The following holds.
(*) Γ_i is a proper latter segment of Γ_j for $0 \leqslant j < i \leqslant n$.
Here, we say that Σ is a latter segment of Γ when $\Gamma \equiv \Delta, \Sigma$.

The outline of the proof of the main theorem was obtained by Nagayama. However due to the lack of the consideration on LBB', this contained an essential gap. With the cut-elimination theorem on LBB', now we can successfully prove the main theorem.

THEOREM 4.3 *For any formula α, the formula $\alpha \rightarrow \alpha$ is not provable in BB'.*

PROOF. Suppose otherwise. Let A be the set of all formulas β such that $\beta \rightarrow \beta$ is provable in BB'. Then A is nonempty. Let α be a formula in A whose length is minimum among formulas in A. We consider a cut-free LBB' proof of $\alpha, \Gamma \Rightarrow p$, where $\alpha \equiv \Gamma \rightarrow p$. We divide the proof into the following three cases according to the last inference of the proof:

(Case 1) the last inference is ($\rightarrow$ left),

(Case 2) the last inference is ($\rightarrow$ left 1),

(Case 3) the last inference is ($\rightarrow$ left 2).

We will show here only for Case 1, however arguments for Case 2 and Case 3 are essentially the same.

In Case 1, the last part of proof would be either one of the following type:

Subcase 1.1

$$\frac{\alpha, \Sigma \Rightarrow \beta \quad \Delta, \gamma, \Theta \Rightarrow p}{\alpha, \Delta \circ (\beta \rightarrow \gamma) \circ \Sigma, \Theta \Rightarrow p} (\rightarrow \text{left}),$$

where $\Gamma \equiv \Delta \circ (\beta \rightarrow \gamma) \circ \Sigma, \Theta$ and $\Sigma \neq \emptyset$.

Subcase 1.2

$$\frac{\Sigma \Rightarrow \beta \quad \alpha, \Delta, \gamma, \Theta \Rightarrow p}{\alpha, \Delta \circ (\beta \rightarrow \gamma) \circ \Sigma, \Theta \Rightarrow p} (\rightarrow \text{left}),$$

where $\Gamma \equiv \Delta \circ (\beta \rightarrow \gamma) \circ \Sigma, \Theta$.

Subcase 1.3

$$\frac{\Sigma_1 \Rightarrow \alpha_1 \quad \Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p}{\alpha, \Delta_1 \circ \Sigma_1, \Gamma_1 \Rightarrow p} (\rightarrow \text{left}),$$

where $\Gamma \equiv \Delta_1 \circ \Sigma_1, \Gamma_1$ and $\alpha \equiv \alpha_1 \rightarrow \overline{\alpha_2}$.

Our goal is to construct a shorter formula δ than α, such that both $\alpha \Rightarrow \delta$ and $\delta \Rightarrow \alpha$ are provable in LBB'. So, we would obtain that $\delta \Rightarrow \delta$ is provable in LBB', which contradicts to the minimality of α.

We argue it according to the subcases above.

Subcase 1.1. In this case, we know that the sequent $\alpha, \Sigma \Rightarrow \beta$ is provable in LBB'. Let δ be $\Sigma \to \beta$. Clearly δ is shorter than α, and $\alpha \Rightarrow \delta$ is provable in LBB'. Now we consider the proof whose last part is of the form

$$\cfrac{\cfrac{\Delta, \gamma, \Theta \Rightarrow p}{\Delta \circ (\beta \to \gamma, \beta), \Theta \Rightarrow p}\ (\to \text{left } 1)}{\delta, \Delta \circ (\beta \to \gamma) \circ \Sigma, \Theta \Rightarrow p}\ (\to \text{left } 1).$$

This proof shows that $\delta \Rightarrow \alpha$ is provable in LBB'.

Subcase 1.2. The same argument as that in Subcase 1.1 proves the claim.

Subcase 1.3. In this case, the argument is not so simple as cases discussed in the above. We need Nagayama's normalization theorem for $LBB'I$. First, we consider a normalized $LBB'I$ proof w.r.t. $\overline{\alpha_2}$ of $\Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p$. Since $\Sigma_1 \neq \emptyset$, $|\Gamma_1| \leqslant n-1$. If the normalized proof has a tail whose length is 1, then $|\Gamma_1| = n-1$. This is because Γ_i is a proper latter segment of Γ_j for $1 \leqslant j < i \leqslant n$. This would imply $\Delta_1 = \emptyset$, and as a result, $\Sigma_1 \equiv \alpha_1$ by the fact that $\Sigma_1, \Gamma_1 \equiv \alpha_1, \cdots, \alpha_n$. However, this would imply $\alpha_1 \Rightarrow \alpha_1$ were provable in LBB', which contradicts to the minimality of the α. Thus there is a tail in the normalized proof w.r.t. $\overline{\alpha_2}$ of $\Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p$ has a tail whose length is greater than 1. According to the last part of the normalized proof, we divide Subcase 1.3 into the following three cases with $\alpha_i \equiv \beta \to \gamma$:

Subcase 1.3.1

$$\frac{\Theta \Rightarrow \beta \quad \Pi, \gamma, \Lambda, \overline{\alpha_2}, \Gamma_1 \Rightarrow p}{\Pi \circ (\alpha_i) \circ \Theta, \Lambda, \overline{\alpha_2}, \Gamma_1 \Rightarrow p}\ (\to \text{left}),$$

where $\Delta_1 \equiv \Pi \circ (\alpha_i) \circ \Theta, \Lambda$.

Subcase 1.3.2

$$\frac{\Theta \Rightarrow \beta \quad \Pi, \overline{\alpha_2}, \Lambda, \gamma, \Phi \Rightarrow p}{(\Pi, \overline{\alpha_2}, \Lambda) \circ (\alpha_i) \circ \Theta, \Phi \Rightarrow p}\ (\to \text{left}),$$

where $\Delta_1, \overline{\alpha_2}, \Gamma_1 \equiv (\Pi, \overline{\alpha_2}, \Lambda) \circ (\alpha_i) \circ \Theta, \Phi$.

Subcase 1.3.3

$$\frac{\Theta, \overline{\alpha_2}, \Pi \Rightarrow \beta \quad \Lambda, \gamma, \Phi \Rightarrow p}{\Lambda \circ (\alpha_i) \circ (\Theta, \overline{\alpha_2}, \Pi), \Phi \Rightarrow p}\ (\to \text{left}),$$

where $\Delta_1, \overline{\alpha_2}, \Gamma_1 \equiv \Lambda \circ (\alpha_i) \circ (\Theta, \overline{\alpha_2}, \Pi), \Phi$.

Finally we construct a shorter formula δ than α claimed above. We demonstrate the construction for Subcase 1.3.3, however arguments for Subcase 1.3.1 and Subcase 1.3.2 are essentially same. First, consider the proof of which the last part is of the form

$$\frac{\Sigma_1 \Rightarrow \alpha_1 \quad \Theta, \overline{\alpha_2}, \Pi \Rightarrow \beta}{\alpha, \Theta \circ \Sigma_1, \Pi \Rightarrow \beta}.$$

Let δ be $\Theta \circ \Sigma_1 \to \Pi \to \beta$. Clearly δ is shorter than α, and the above proof shows that $\alpha \Rightarrow \delta$ is provable in $LBB'I$. Since $\alpha \Rightarrow \delta$ is non-trivial, Powers and Dwyer's theorem implies that $\alpha \Rightarrow \delta$ is provable in LBB'.

Now we consider the proof of which the last part is of the form

$$\frac{\delta, \Theta \circ \Sigma_1, \Pi \Rightarrow \beta \quad \Lambda, \gamma, \Phi \Rightarrow p}{\delta, \Lambda \circ (\alpha_i) \circ (\Theta \circ \Sigma_1, \Pi), \Phi \Rightarrow p} (\to \text{left}).$$

Because the sequent $\delta, \Theta \circ \Sigma_1, \Pi \Rightarrow \beta$ is trivial, it is provable in $LBB'I$. As $\Lambda \circ (\alpha_i) \circ (\Theta \circ \Sigma_1, \Pi), \Phi \equiv \Delta_1 \circ \Sigma_1, \Gamma_1$, this shows that $\delta \Rightarrow \alpha$ is provable in $LBB'I$. Since $\delta \Rightarrow \alpha$ is non-trivial, Powers and Dwyer's theorem implies that $\delta \Rightarrow \alpha$ is provable in LBB'. Therefore $\delta \Rightarrow \delta$ is provable in LBB'. But this contradicts the minimality of the length of α. ∎

By Theorem 3.14 and Theorem 4.3, we have Theorem 1.2. As a trivial formula is not provable in the logic BB', of course it is not provable in the logic B. So, we have Theorem 1.1 by Theorem 2.8. It follows from Theorem 1.1 and Theorem 2.7 that the logic B is decidable. Therefore, we have Theorem 1.3 by Theorem 2.7 and Theorem 3.13.

Acknowledgement

The author would like to thank M. Nagayama for sending him her manuscript [6]. The last section of the present paper would never exist without Nagayama's paper. The author also wishes to thank J. Slaney, R.K. Meyer, H. Ono, E.P. Martin, M.W. Bunder and referees for useful discussions with them and for pointing out to him errors, some of which are serious, in the previous versions. The author thanks M. Tatsuta for his TEX's style file proof.sty.

References

[1] A. R. Anderson and N. D. Belnap Jr, **Entailment: The logic of relevance and necessity,** Volume 1, Princeton university Press, 1975.

[2] A. R. ANDERSON, N. D. BELNAP JR. and J. M. DUNN, **Entailment: The logic of relevance and necessity,** Volume 2, Princeton university Press, 1992.

[3] G. GENTZEN, *Untersuchungen über das logische Schliessen,* **Mathematische Zeitschrift 39** 1935, pp. 176–210, 405–431, Translated in [9].

[4] Y. KOMORI, *Predicate logics without the structure rules,* **Studia Logica 45** 1986, pp. 393–404.

[5] E. P. MARTIN and R. K. MEYER, *Solution to the $P - W$ problem,* **Journal of Symbolic Logic 47** 1982, pp. 869–887.

[6] M. NAGAYAMA, *Syntactic solution to the $P-W$ problem,* 1993, submitted to **Journal of Symbolic Logic.**

[7] H. ONO and Y. KOMORI, *Logics without the contraction rule,* **Journal of Symbolic Logic 50** 1985, pp. 169–201.

[8] R. ROUTLEY, R. K. MEYER, V. PLUMWOOD and R. T. BRADY, **Relevant logics and their rivals 1,** Ridgeview, 1982.

[9] M. E. SZABO, **The collected papers of Gerhard Gentzen,** North-Holland, Amsterdam 1969.

[10] G. TAKEUTI, *Proof theory,* **Studies in Logic and the Foundations of Mathematics,** Volume 81, 1975, North-Holland, Amsterdam.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
SHIZUOKA UNIVERSITY
OHYA SHIZUOKA 422 JAPAN