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The Variety Generated by BCC-Algebras Is Finitely Based

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Dedicated to Professor Kentaro Murata on his 60th birthday

In [2], we have proved that the class of all BCC-algebras is not a variety. In this note, developing the method in [2], we shall show that *the variety generated by BCC-algebras*, that is, the smallest variety containing the class of all BCC-algebras, is finitely based. (For the definitions and notations undefined here, see the reference [1].)

A BCC-algebra is an algebra $\mathbf{A} = \langle A; \rightarrow, 1\rangle$ of type $\langle 2,0\rangle$ such that for every $x, y, z \in A$ the following conditions are satisfied:

(1) $(y \rightarrow z) \rightarrow (x \rightarrow y) \rightarrow x \rightarrow z = 1,$

(2) $x \rightarrow x = 1,$

(3) $x \rightarrow 1 = 1,$

(4) $1 \rightarrow x = x,$

(5) *if* $x \rightarrow y = 1$ *and* $y \rightarrow x = 1$, *then* $x = y$.

We have the axiom system of BCK-algebras (but dual form), if we exchange (1) for $(x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$. We adopt the convention requiring terms with lacking parenthesis to be associated to the right.

We define *Gentzen-type system* **LC**. (This system is slightly defferent from the system in [2]. But, of course, thay are mutually equivalent.) In the following, Γ, Δ, Σ denote finite (possibly empty) sequences of terms separated by commas. The followings are axioms and rules of inference of **LC**.

Axioms :

$\Gamma, \alpha, \Delta \Rightarrow \alpha$ (for any variable α),

$\Gamma \Rightarrow 1.$

Rules of inference:

$$\text{cut} \ : \ \frac{\Gamma \Rightarrow t \qquad \Sigma, t, \Delta \Rightarrow s}{\Sigma, \Gamma, \Delta \Rightarrow s}$$

$$\Rightarrow\rightarrow \ : \ \frac{\Gamma, s \Rightarrow t}{\Gamma \Rightarrow s \rightarrow t} \qquad \rightarrow\Rightarrow \ : \ \frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow u}{\Sigma, s \rightarrow t, \Gamma, \Delta \Rightarrow u.}$$

We write $\Gamma \vdash t$ if the sequent $\Gamma \Rightarrow t$ is provable in **LC**. We write $\Gamma \nvdash t$ if the sequent $\Gamma \Rightarrow t$ is not provable in **LC**. If $s \vdash t$ and $t \vdash s$, then we write $s \Leftrightarrow t$.

Theorem 1 ([2]). *For any terms* s *and* t*;*
(i) $\vdash t$ *if and only if* $t = 1$ *is satisfied in all BCC-algebras,*
(ii) $s \Leftrightarrow t$ *if and only if* $s = t$ *is satisfied in all BCC-algebras.*

Theorem 2 (*Cut Elimination Theorem* [2]). *If* $\Gamma \Rightarrow t$ *is provable in* **LC**, *then it is provable without a cut in* **LC**.

The following lemma is useful and fundamental.

Lemma 3 (*Fundamental Lemma*). *Let* Γ *be a finite sequence of variables and* α *be a variable. If* $\Gamma, s \to t, \Delta \vdash \alpha$ *and* $\Gamma, \Delta \nvdash \alpha$, *then there exist sequences* Δ_1 *and* Δ_2 *such that* $\Delta_1 \vdash s$, $\Gamma, t, \Delta_2 \vdash \alpha$ *and* $\Delta = \Delta_1, \Delta_2$.

Proof. We prove this lemma by induction on the length of cut-free **LC**-proof of $\Gamma, s \to t, \Delta \Rightarrow \alpha$. By $\Gamma, \Delta \nvdash \alpha$, $\Gamma, s \to t, \Delta \Rightarrow \alpha$ is not an axiom.

Case 1. The principal term of the last inference is $s \to t$, that is, the last inference is

$$\frac{\Delta_1 \Rightarrow s \qquad \Gamma, t, \Delta_2 \Rightarrow \alpha}{\Gamma, s \to t, \Delta_1, \Delta_2 \Rightarrow \alpha} \quad (\Delta = \Delta_1, \Delta_2)$$

In this case, this lemma obviously holds.

Case 2. The principal term of the last inference is contained in Δ, that is, the last inference is

$$\frac{\Sigma_1 \Rightarrow u \qquad \Gamma, s \to t, \Sigma_2, v, \Sigma_3 \Rightarrow \alpha}{\Gamma, s \to t, \Sigma_2, u \to v, \Sigma_1, \Sigma_3 \Rightarrow \alpha} \quad (\Delta = \Sigma_2, u \to v, \Sigma_1, \Sigma_3)$$

By $\Gamma, \Delta \nvdash \alpha$, $\Gamma, \Sigma_2, v, \Sigma_3 \nvdash \alpha$. By induction hypothesis, there exist sequences Π_1 and Π_2 such that $\Pi_1 \vdash s$, $\Gamma, t, \Pi_2 \vdash \alpha$ and $\Sigma_2, v, \Sigma_3 = \Pi_1, \Pi_2$. Suppose that v is contained in Π_1. Then, there exists a sequence Λ such that $\Pi_1 = \Sigma_2, v, \Lambda$. By $\Sigma_1 \vdash u$ and $\Sigma_2, v, \Lambda \vdash s$, $\Sigma_2, u \to v, \Sigma_1, \Lambda \vdash s$. Therefore, we have that $\Delta_1 \vdash s$, $\Gamma, t, \Delta_2 \vdash \alpha$ and $\Delta = \Delta_1, \Delta_2$ if we put $\Delta_1 = \Sigma_2, u \to v, \Sigma_1, \Lambda$ and $\Delta_2 = \Pi_2$. Suppose that v is contained in Π_2. Then, there exists a sequence Λ such that $\Pi_2 = \Lambda, v, \Sigma_3$. By $\Sigma_1 \vdash u$ and $\Gamma, t, \Lambda, v, \Sigma_3 \vdash \alpha$, $\Gamma, t, \Lambda, u \to v, \Sigma_1, \Sigma_3 \vdash \alpha$. It completes the proof that we put $\Delta_1 = \Pi_1$ and $\Delta_2 = \Lambda, u, \to v, \Sigma_1, \Sigma_3$. **Q. E. D.**

We regard $t_m, t_{m+1}, \cdots\cdots, t_n$ as the empty sequence if $n < m$.

Lemma 4. *If* $t_1 \to t_2 \to \cdots \to t_n \to \alpha$, $t_k, t_{k+1}, \cdots, t_n \vdash \alpha$ *and* $t_k, t_{k+1}, \cdots, t_n \nvdash \alpha$ *(*$1 \leq k \leq n+1$ *and* α *is a variable), then* $\vdash t_l$ *for any* l *such that* $1 \leq l \leq k-1$.

Proof. If $k = 1$, then this lemma holds obviously because there does not exist l such that $1 \leq l \leq k-1$. Suppose that $2 \leq k \leq n+1$. We prove this lemma by induction on n. By Lemma 3, there exists a natural number m ($k-1 \leq m \leq n$) such that $t_k, t_{k+1}, \cdots, t_m \vdash t_1$ and $t_2 \to \cdots \to t_n \to \alpha, t_{m+1}, t_{m+2}, \cdots, t_n \vdash \alpha$. By $t_k, t_{k+1}, \cdots, t_n \nvdash \alpha$, we have $t_{m+1}, t_{m+2}, \cdots, t_n \nvdash \alpha$. By induction hypothesis, $\vdash t_i$ for any i ($2 \leq i \leq m$). By $t_k, t_{k+1}, \cdots, t_m \vdash t_1$ ($k \geq 2$) and $\vdash t_i$ for any i ($2 \leq i \leq m$), we have $\vdash t_1$. Hence, $\vdash t_l$ for any l ($1 \leq l \leq k-1$). **Q. E. D.**

Lemma 5. *If* $s \to t \Leftrightarrow u \to v$, *then at least one of the following four conditions holds ; (A)* $s \Leftrightarrow u$ *and* $t \Leftrightarrow v$, *(B)* $\vdash s$ *and* $t \Leftrightarrow u \to v$, *(C)* $\vdash u$ *and* $s \to t \Leftrightarrow u$, *(D)* $\vdash s \to t$ *and* $\vdash u \to v$.

Proof. Let t and v be $t_1 \to t_2 \to \cdots \to t_m \to \alpha$ and $v_1 \to v_2 \to \cdots \to v_n \to \beta$ ($m, n \geq 0$ and α and β are variables), respectively. By $s \to t \vdash u \to v$, $s \to t, u, v_1, v_2, \cdots, v_n \vdash \beta$. If $u, v_1, v_2, \cdots, v_n \vdash \beta$, then (D) holds. Suppose $u, v_1, v_2, \cdots, v_n \nvdash \beta$. By Lemma 3, either $\vdash s$ and $t, u, v_1, v_2, \cdots, v_n \vdash \beta$, or there exists l ($0 \leq l \leq n$) such that $u, v_1, v_2, \cdots, v_l \vdash s$ and $t, v_{l+1}, v_{l+2}, \cdots, v_n \vdash \beta$. In the former case, (B) holds. Consider the latter case. By $u \to v \vdash s \to t$, $u \to v, s, t_1, t_2, \cdots, t_m \vdash \alpha$. We can suppose $s, t_1, t_2, \cdots, t_m \nvdash \alpha$ because otherwise (D) holds. By Lemma 3, either $\vdash u$ and $v, s, t_1, t_2, \cdots, t_m \vdash \alpha$, or there exists k ($0 \leq k \leq m$) such that $s, t_1, t_2, \cdots, t_k \vdash u$ and $v, t_{k+1}, t_{k+2}, \cdots, t_m \vdash \alpha$. If $\vdash u$, then (C) holds. By $t, v_{l+1}, v_{l+2}, \cdots, v_n \vdash \beta$, $t_{k+1} \to t_{k+2} \to \cdots \to t_m \to \alpha, v_{l+1}, v_{l+2}, \cdots, v_n \vdash \beta$. Hence, by $v, t_{k+1}, t_{k+2}, \cdots, t_m \vdash \alpha$, $v, v_{l+1}, v_{l+2}, \cdots, v_n \vdash \beta$. Therefore, by Lemma 4, $\vdash v_i$ for any i ($1 \leq i \leq l$). Hence, by $u, v_1, v_2, \cdots, v_l \vdash s$, $u \vdash s$. Similarly, we have $s \vdash u$. We have proved that (A) holds. **Q. E. D.**

The degree of a term t (denoted by deg (t)) is the number of occurrences of the symbol $\to$ in t.

Theorem 6. *The variety generated by the class of all BCC-algebras is finitely based. The equational base is* $\{(1), (2), (3), (4)\}$, *where (1), (2), (3) and (4) are the axioms of BCC-algebras in this note p. 13.*

Proof. We write $\mapsto s = t$ if $s = t$ is provable from $\{(1), (2), (3), (4)\}$. It suffices to show that, for any s and t, $\mapsto s = t$ if $s \Leftrightarrow t$. By induction on deg (s) + deg (t). If deg (s) + deg $(t) = 0$, then s and t are the same variables. Hence $\mapsto s = t$. Suppose deg (s) + deg $(t) \geq 1$.

Case 1: deg$(s) = 0$, that is, s is a variable.

Let t be $t_1 \to t_2$. By $t_1 \to t_2 \vdash s$ and Lemma 4 (because s is a variable), we have $\vdash t_1$. Therefore $s \Leftrightarrow t_2$ and $\mapsto t_1 = 1$. By induction hypothesis, we have $\mapsto s = t_2$. Hence, by $\mapsto t_1 = 1$, we have $\mapsto s = t$. (Note that $\vdash s$ iff $\mapsto s = 1$ for any s.)

Case 2: deg$(t) = 0$. Similaly to Case 1.

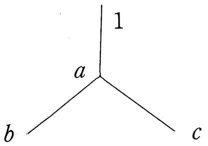
Case 3: deg$(s) \geq 1$ and deg$(t) \geq 1$.

Let s and t be $s_1 \to s_2$ and $t_1 \to t_2$, respectively. By Lemma 5, at least one of the following conditions holds; (A) $s_1 \Leftrightarrow t_1$ and $s_2 \Leftrightarrow t_2$, (B) $\vdash s_l$ and $s_2 \Leftrightarrow t_1 \to t_2$, (C) $\vdash t_1$ and $s_1 \to s_2 \Leftrightarrow t_2$ or, (D) $\vdash s_1 \to s_2$ and $\vdash t_1 \to t_2$. We prove it only in the case (A). In other cases, the proof is easy and similar. By induction hypothesis, we have that $\mapsto s_1 = t_1$ and $\mapsto s_2 = t_2$. Hence we have $\mapsto s = t$. **Q. E. D.**

We can obtain an axiom system for BCK-algebras by adding the identity (6) $x \to y \to z = y \to x \to z$ to our axiom system for BCC-algebras. But $\{(1), (2), (3), (4), (6)\}$ is not an equational base of the variety generated by all BCK-algebras.

Theorem 7. *The identity $x \to y = ((x \to y) \to y) \to y$ is satisfied in all BCK-algebras, but the identity is not provable from $\{(1), (2), (3), (4), (6)\}$.*

Proof. We prove only that the identity is not provable from $\{(1), (2), (3), (4), (6)\}$. Consider the following Hasse diagram. We define the function $\to$ on $\{1, a, b, c\}$ as follows:



for any x, $1 \to x = x$,
if $x \leq y$, then $x \to y = 1$,
$b \to c = c \to b = 1$, $a \to b = c$ and $a \to c = b$.

In this algebra, all of (1), (2), (3), (4) and (6) are satisfied. But, in the identity, we substitute a and b for x and y respectively. We have that the left side $= c$ and the right side $= b$. Hence the identity is not satisfied in this algebra. Therefore the identity is not provable from $\{(1), (2), (3), (4), (6)\}$. Q. E. D.

Problem. *Is the variety generated by all BCK-algebras finitely based?*

References

[1] G. Grätzer, Universal Algebra (second edition), Springer-Verlag New York Inc., 1979.
[2] Y. Komori, The class of BCC-algebras is not a variety, to appear.

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