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THE WORD PROBLEM FOR FREE BCI-ALGEBRAS IS DECIDABLE

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ABSTRACT. We introduce two Gentzen-type sequent calculi: one reflecting the structure of the BCI-algebra, and the other enjoying the cut-elimination theorem. We prove the equivalence of the two systems, and as its corollary we show the decidability of the word problem for free BCI-algebras.

A *BCI-algebra* is an algebra[1] $A = \langle A; \to, 1\rangle$ of type $\langle 2, 0\rangle$ such that for every x, y, z in A the following conditions are satisfied:

(1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

(2) $x \to ((x \to y) \to y) = 1$;

(3) $x \to x = 1$;

(4) if $x \to y = y \to x = 1$ then $x = y$;

(5) if $1 \to x = 1$ then $x = 1$.

A *BCI*-algebra was originally defined by Iséki [3], [4] in exact dual form of the definition above. We prefer ours, since it reflects the relation to logic and other algebras related to logic; for instance, one can observe from the definition that a Boolean algebra is a special case of BCI-algebras.

A *term* is an expression built up as usual from variables, a constant symbol "1" and a binary operation symbol "$\to$". If $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta$ are terms ($n \geq 0$), then an expression $\alpha_1, \alpha_2, \cdots, \alpha_n \Rightarrow \beta$ is called a *sequent.* We will use letters $x, y, \cdots$ for variables, $\alpha, \beta, \cdots$ for terms and $\Gamma, \Delta, \cdots$ for finite (possibly empty) sequences of terms. $X \equiv Y$ will mean X is exactly the same expression as Y. Parentheses will be omitted in such a way that, for example, $\alpha \to \beta \to \gamma \to \delta$ denotes $\alpha \to (\beta \to (\gamma \to \delta))$.

We now define a sequent calculus *BCI-pw*, which is a subsystem of Gentzen's *LJ* (see [8] for *LJ*).

Axioms of *BCI-pw* :

$$\alpha \Rightarrow \alpha \quad \text{and} \quad \Rightarrow 1$$

Inference rules of *BCI-pw* :

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma} \ \text{exchange}$$

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[1]For the definitions of algebras and related notions, see [2].

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ \text{cut}$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \to \text{right}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Delta \Rightarrow \gamma} \to \text{left}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Rightarrow \beta}{\beta, \Gamma \Rightarrow \alpha}\ \text{provably weakening}$$

Example of proof in BCI-pw.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \Rightarrow x \quad \dfrac{\dfrac{x \to y \Rightarrow x \to y \quad x \Rightarrow x}{(x \to y) \to x, x \to y \Rightarrow x} \to l.}{x \to y, (x \to y) \to x \Rightarrow x}\, ex.}{x \to x \to y, x, (x \to y) \to x \Rightarrow x} \to l.}{x, x \to x \to y, (x \to y) \to x \Rightarrow x}\, ex.}{x \to x \to y, (x \to y) \to x \Rightarrow x \to x} \to r. \quad \dfrac{\dfrac{z \Rightarrow z}{\Rightarrow z \to z} \to r. \quad \dfrac{x \Rightarrow x}{\Rightarrow x \to x} \to r.}{x \to x \Rightarrow z \to z}\, p.w.}{x \to x \to y, (x \to y) \to x \Rightarrow z \to z}\, cut$$

Theorem 1. (1) *$\alpha = 1$ is satisfied in all BCI-algebras if and only if $\Rightarrow \alpha$ is provable in BCI-pw.*

(2) *$\alpha = \beta$ is satisfied in all BCI-algebras if and only if both $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ are provable in BCI-pw.*

Proof. (If part of 1) By induction on the length of a proof in *BCI-pw* , we can prove that if $\beta_1, \beta_2, \cdots, \beta_n \Rightarrow \alpha$ is provable in *BCI-pw* then $\beta_1 \to \beta_2 \to \cdots \to \beta_n \to \alpha = 1$ is satisfied in all *BCI*-algebras. (To show this, we use some properties of *BCI*-algebras shown in [3], [4]; for example, $x \to y \to z = y \to x \to z$.)

(Only if part of 1) The Lindenbaum algebra (see [1]) of *BCI-pw* is a *BCI*-algebra. Hence, if $\Rightarrow \alpha$ is not provable in *BCI-pw* then $\alpha = 1$ is not satisfied in some *BCI*-algebra.

(2) Easily shown by (1). (Note that in a *BCI*-algebra $\alpha = \beta$ is equivalent to $\alpha \to \beta = \beta \to \alpha = 1$.) ∎

Theorem 2. *The cut-elimination theorem does not hold for BCI-pw , i.e. there is a sequent which is provable in BCI-pw but is not provable without cut.*

Proof. For example, $x \to x \to y, (x \to y) \to x \Rightarrow z \to z$ is provable as above, but is not provable without cut. ∎

For any term α, we define multisets[2] $pos(\alpha)$ and $neg(\alpha)$ of variables, inductively as follows:

(1) $pos(x) = \{x\}, neg(x) = pos(1) = neg(1) = \{\}$;

(2) $pos(\alpha \to \beta) = neg(\alpha) \cup pos(\beta), neg(\alpha \to \beta) = pos(\alpha) \cup neg(\beta)$.

Moreover, we define $pos(\alpha_1, \alpha_2, \cdots, \alpha_n) = pos(\alpha_1) \cup pos(\alpha_2) \cup \cdots \cup pos(\alpha_n), neg(\alpha_1, \alpha_2, \cdots, \alpha_n) = neg(\alpha_1) \cup neg(\alpha_2) \cup \cdots \cup neg(\alpha_n), pos(\Gamma \Rightarrow \alpha) = neg(\Gamma) \cup pos(\alpha), neg(\Gamma \Rightarrow \alpha) = pos(\Gamma) \cup neg(\alpha)$. For example,

$$pos(x \to y \to z, (x \to y) \to z \Rightarrow x) = \{x, x, y, y\},$$
$$neg(x \to y \to z, (x \to y) \to z \Rightarrow x) = \{x, z, z\}.$$

[2] A multiset is a set which can contain the same elements several times.

We say Γ is *balanced* if $pos(\Gamma) = neg(\Gamma)$. Also, we say $\Gamma \Rightarrow \alpha$ is *balanced* if $pos(\Gamma \Rightarrow \alpha) = neg(\Gamma \Rightarrow \alpha)$.

Now, we introduce a Gentzen-type sequent calculus *BCI-bw* .

Axioms of *BCI-bw* :

$$\alpha \Rightarrow \alpha \quad \text{and} \quad \Rightarrow 1$$

Inference rules of *BCI-bw* : exchange, cut, $\rightarrow$ right, $\rightarrow$ left, and

$$\frac{\Gamma \Rightarrow \alpha}{\Delta, \Gamma \Rightarrow \alpha} \text{ balanced weakening}$$

where Δ is balanced.

Example of proof in BCI-bw

$$\cfrac{\cfrac{z \Rightarrow z}{\Rightarrow z \rightarrow z} \rightarrow r.}{x \rightarrow x \rightarrow y, (x \rightarrow y) \rightarrow x \Rightarrow z \rightarrow z} \; b.w.$$

Theorem 3. *A sequent is provable in BCI-pw if and only if it is provable in BCI-bw .*

To prove this we need some lemmas.

Lemma 4. *For any term α, there exist sequences Π_α and Σ_α which satisfies the following conditions :*

(1) *Both $\alpha, \Pi_\alpha \Rightarrow 1$ and $\Sigma_\alpha \Rightarrow \alpha$ are balanced and provable in BCI-pw ;*

(2) *each term in $\Pi_\alpha, \Sigma_\alpha$ is of the form x or $x \rightarrow 1$ (x is a variable).*

Proof. By induction on the length of α.

If $\alpha \equiv 1$, then Π_α and Σ_α are empty.

If $\alpha \equiv y$ (y is a variable), then $\Pi_\alpha \equiv y \rightarrow 1$ and $\Sigma_\alpha \equiv y$.

If $\alpha \equiv \beta \rightarrow \gamma$, then $\Pi_\alpha \equiv \Sigma_\beta, \Pi_\gamma$ and $\Sigma_\alpha \equiv \Pi_\beta, \Sigma_\gamma$ where $\Pi_\beta, \Sigma_\beta, \Pi_\gamma, \Sigma_\gamma$ are the sequences given by the induction hypotheses for β and γ. ∎

Lemma 5. *If Δ is a non-empty balanced sequence and $\Gamma \Rightarrow \alpha$ is provable in BCI-pw then $\Delta, \Gamma \Rightarrow \alpha$ is provable in BCI-pw .*

Proof. Let $\Delta \equiv \delta_1, \delta_2, \cdots, \delta_n$ ($n \geq 1$). By Lemma 4, there exist Π_i for $i = 1, 2, \cdots, n$ such that:

(1) $\delta_i, \Pi_i \Rightarrow 1$ is balanced and provable in *BCI-pw* ;

(2) each term in Π_i is of the form x or $x \rightarrow 1$.

Let $\Pi \equiv \Pi_1, \Pi_2, \cdots, \Pi_n$. Then $\Delta, \Pi \Rightarrow 1$ is provable in *BCI-pw* as follows:

(3)

$$\cfrac{\overset{(1)}{\delta_1, \Pi_1 \Rightarrow 1} \qquad \cfrac{\overset{(1)}{\delta_2, \Pi_2 \Rightarrow 1} \quad \Rightarrow 1}{1, \delta_2, \Pi_2 \Rightarrow 1}\, p.w.}{\delta_1, \Pi_1, \delta_2, \Pi_2 \Rightarrow 1}\, cut$$

$$\vdots$$

$$\Delta, \Pi \Rightarrow 1$$

On the other hand, Π is a permutation of

$$x_1, x_1 \rightarrow 1, x_2, x_2 \rightarrow 1, \cdots, x_m, x_m \rightarrow 1,$$

since Δ and $\Delta, \Pi \Rightarrow 1$ are balanced. Then $\Pi \Rightarrow 1$ is provable in *BCI-pw* as follows:

(4)

$$\dfrac{x_2 \Rightarrow x_2 \qquad \dfrac{\dfrac{x_1 \Rightarrow x_1 \quad 1 \Rightarrow 1}{x_1 \to 1, x_1 \Rightarrow 1}\,{\to}\, l. \qquad \Rightarrow 1}{1, x_1 \to 1, x_1 \Rightarrow 1}\,p.w.}{x_2 \to 1, x_2, x_1 \to 1, x_1 \Rightarrow 1}\,{\to}\, l.$$

$$\vdots$$

$$\Pi \Rightarrow 1$$

Hence, if $\Gamma \Rightarrow \alpha$ is provable in *BCI-pw* , then $\Delta, \Gamma \Rightarrow \alpha$ is provable as follows:

$$\dfrac{\begin{array}{c}(3)\\ \Delta, \Pi \Rightarrow 1 \\ \vdots \\ \Delta \Rightarrow \pi_1 \to \pi_2 \to \cdots \to \pi_{2m} \to 1\end{array} \qquad \dfrac{\Gamma \Rightarrow \alpha \qquad \begin{array}{c}(4)\\ \Pi \Rightarrow 1 \\ \vdots \\ \Rightarrow \pi_1 \to \pi_2 \to \cdots \to \pi_{2m} \to 1\end{array}}{\pi_1 \to \pi_2 \to \cdots \to \pi_{2m} \to 1, \Gamma \Rightarrow \alpha}\,p.w.}{\Delta, \Gamma \Rightarrow \alpha}\,cut$$

where $\pi_1, \pi_2, \cdots, \pi_{2m} \equiv \Pi$. ∎

Lemma 6. *If a sequent is provable in BCI-bw then it is balanced.*

Proof. By induction on the length of proof in *BCI-bw* . ∎

Proof of Theorem 3. By Lemmas 5 and 6, and induction on the length of proofs in *BCI-pw* and *BCI-bw* . ∎

We can show the cut-elimination theorem and the decidability for *BCI-bw* as those of *LJ* shown in [8].

Theorem 7. *If a sequent is provable in BCI-bw , then it is also provable in BCI-bw without cut.*

Theorem 8. *The problem whether given sequents are provable in BCI-bw or not is decidable.*

Then we obtain the following corollary:

Corollary 9. *The word problem for free BCI-algebras is decidable.*

Proof. By Theorems 1, 3 and 8. ∎

Remarks. 1. If we extend *BCI-pw* by replacing provably weakening by

$$\dfrac{\Gamma \Rightarrow \alpha}{\beta, \Gamma \Rightarrow \alpha}\,\text{weakening}$$

(β is an arbitrary term), then we obtain a sequent calculus called *LKK* ([6]) or L_{BCK} ([7]). The results which correspond to Theorems 1, 7 and 8, and Corollary 9, for *LKK* and the *BCK*-algebra (see [5] for *BCK*-algebra) are shown in [6].

2. The converse of Lemma 6 does not hold. For example,

$$\Rightarrow (((x \to 1) \to 1) \to (x \to 1) \to y) \to y$$

is balanced and is provable in *LKK*, but is not provable in *BCI-bw* .

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