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A New Semantics for Intuitionistic Predicate Logic

Abstract. The main part of the proof of Kripke's completeness theorem for intuitionistic logic is Henkin's construction. We introduce a new Kripke-type semantics with semilattice structures for intuitionistic logic. The completeness theorem for this semantics can be proved without Henkin's construction.

In the proofs of Gödel's completeness theorem for classical logic and Kripke's completeness theorem for intuitionistic logic, Henkin's construction plays a large part. In Henkin's construction, for any theory T and any formula α such that $\alpha \notin T$, we construct a theory T' satisfying the following three conditions: (1) $T \subset T'$ and $\alpha \notin T'$, (2) $\beta \vee \gamma \in T' \Rightarrow \beta \in T'$ or $\gamma \in T'$, (3) $\exists x\beta(x) \in T' \Rightarrow$ for some constant a $\beta(a) \in T'$. Nevertheless, we cannot construct such a theory for the logic L_{BCK} obtained by deleting the contraction rule from Gentzen's LJ. It is closely related with the fact that neither the sequent $\alpha\wedge(\beta\vee\gamma)\rightarrow(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$ nor the sequent $\alpha\wedge\exists x\beta(x)\rightarrow\exists x(\alpha\wedge\beta(x))$ can be proved in it. So, in [3] and [7] we have had to prove the completeness theorem for L_{BCK} without Henkin's construction by changing the interpretation of $\vee$ and $\exists$. Through such consideration, we get a new semantics for intuitionistic logic. In our previous papers, [2], [4] and [6], for some intermediate predicate logics we have obtained incompleteness results with respect to Kripke semantics and the algebraic semantics. So, our new semantics may have the new power in investigating the intermediate predicate logics. In this paper, we will show the completeness theorem by using Henkin's construction. But it is possible to prove the completeness theorem without using it. Indeed, we can show the completeness theorem similarly to the completeness theorem for L_{BCA} in [3]. On the other hand, our proof shown in the following will clarify the relationship between our semantics and Kripke's original one.

1. New Kripke frames and Soundness Theorem

We begin with the definition of new Kripke models. Then, we prove the soundness theorem with respect to the models.

Definition 1.1. A structure $\langle M; \infty, \cap\rangle$ is a *∞-distributive semilattice,* if it satisfies the following three conditions:

(1) $\langle M; \cap\rangle$ is a meet-semilattice. (The relation $a\cap b = a$ is denoted by $a \leqslant b$. Clearly, the relation $\leqslant$ is a partial order.)

(2) ∞ is a mapping on M to M satisfying the following condition; for any $a, b \in M$ (i) $\infty(a) \leqslant b \Rightarrow b = \infty(b)$, (ii) $a \leqslant \infty(a)$, (iii) $\infty(a \cap b) = \infty(a) \cap \infty(b)$.

(3) $\langle M;\ \infty,\ \cap \rangle$ is ∞-distributive, that is, for any $a, b, c \in M$, if $a \cap b \leqslant c$ then there exists $a' \in M$ such that $a' \cap b \leqslant c$, $a' \geqslant a$ and $a' \geqslant c \cap \infty(a \cap b)$.

Hereafer, we say that b is an extension of $a \leqslant b$.

DEFINITION 1.2. A structure $\mathbf{M} = \langle M;\ \infty,\ \cap, K, U \rangle$ is a *Kripke frame*, if $\langle M;\ \infty,\ \cap \rangle$ is a ∞-distributive semilattice, K is a subset of M, U is a mapping from K to a family of sets and the following four conditions are satisfied:

(1) for any $a, b \in M$ and any $c \in K$, if $a \cap b \leqslant c$ then a has an extension a' in K such that $a' \cap b \leqslant c$,

(2) for any $a \in K$, $U(a) \neq \varnothing$,

(3) for any $a, b, c \in K$, if $a \cap b \leqslant c$ then $U(a) \cap U(b) \subset U(c)$,

(4) for any $a, b, c \in K$, if $a \cap b \leqslant c$ and $\infty(a) = \infty(b)$, then both a and b have extensions a' and b' in K, resp., such that $a' \cap b' \leqslant c$ and $U(a') \cap U(b') = U(c)$.

In the above definition, K and U are called a *frame subset* of $\mathbf{M}$ (or $\langle M;\ \infty,\ \cap \rangle$) and a *universe function* of $\mathbf{M}$ (or $\langle M;\ \infty,\ \cap, K \rangle$), respectively.

MOTIVATION. Let $\langle M;\ \infty,\ \cap, K, U \rangle$ be a Kripke frame. Let a and b be elements of M. In the proof of the completeness theorem without Henkin's construction, it is as follows. M is a collection of all theories. $\infty(a)$ is the contradictory theory whose language is the same as that of a, that is, $\infty(a)$ consists of all formulas of the language of a. $a = \infty(a)$ if and only if a is a contradictory theory. $a \cap b$ is the intersection of theories a and b. K is equal to M. $U(a)$ is the language of a theory a. In this case, it holds that $K = M$ and $U(\infty(a)) = U(a)$. But it does not generally hold (cf. Section 2).

Let a Kripke frame $\mathbf{M}$ ($= \langle M;\ \infty,\ \cap, K, U \rangle$) be given. Usually, the name of an element u of $U(a)$ is denoted by $\bar{u}$, but we use u instead of $\bar{u}$ for the sake of simplicity. Let L be a first order language without function symbols or constant symbols. The language obtained from L by adding all names of elements of $U(a)$ is denoted by $L(a)$. We regard $L(a)$ as the collection of logical symbols, formulas and so on of $L(a)$. That is to say, if α is a formula, $\alpha \in L(a)$ means that α is a formula of $L(a)$. $\bigcup_{a \in K} L(a)$ is denoted by $L(\mathbf{M})$. The set of all closed formulas of $L(\mathbf{M})$ is denoted by $W(\mathbf{M})$ (sometimes, denoted simply by W). $AW(\mathbf{M})$ (or simply, AW) denotes the set of all closed atomic formulas of $L(\mathbf{M})$. Now, a valuation of $\mathbf{M}$ is a relation between K and $AW(\mathbf{M})$.

DEFINITION 1.3. A pair $\langle \mathbf{M};\ \models \rangle$ is a *Kripke model*, if $\mathbf{M}$ is a Kripke frame and a subset $\models$ of $K \times AW$ ($(a, \alpha) \in \models$ is denoted by $a \models \alpha$) satisfies the following condition (1).

(1) For any $\alpha \in AW(\mathbf{M})$ and any $a, b, c \in K$,

(1–1) $a \models \alpha \Rightarrow \alpha \in L(a)$,

(1–2) $a = \infty(a)$ and $\alpha \in L(a) \Rightarrow a \models \alpha$,

(1–3) $a \models \alpha$ and $b \models \alpha$ and $a \cap b \leqslant c \Rightarrow c \models \alpha$.

Such a relation $\models$ is called a ***valuation*** (or a *forcing*) of **M**. We extend each valuation $\models$ to a relation between K and $W(\mathbf{M})$, inductively as follows. For any $\alpha, \beta \in W(\mathbf{M})$ and any $a \in K$,

(2) $a \models \alpha \supset \beta \Leftrightarrow \forall b \in K(a \leqslant b$ and $b \models \alpha$ imply $b \models \beta)$ and $\alpha, \beta \in L(a)$,

(3) $a \models \alpha \vee \beta \Leftrightarrow \exists b, c \in K\big(b \cap c \leqslant a$ and $b \models \alpha$ and $c \models \beta$ and $\beta \in L(b)$ and $\alpha \in L(c)\big)$,

(4) $a \models \alpha \wedge \beta \Leftrightarrow a \models \alpha$ and $a \models \beta$,

(5) $a \models \neg\alpha \Leftrightarrow \forall b \in K\big(a \leqslant b$ and $b \models \alpha$ imply $b = \infty(b)\big)$ and $\alpha \in L(a)$,

(6) $a \models \forall x\alpha(x) \Leftrightarrow \forall b, c \in K \forall u \in U(b)(a \cap c \leqslant b$ and $c = \infty(c)$ and $\forall x\alpha(x) \in L(c)$ imply $b \models \alpha(u))$ and $\forall x\alpha(x) \in L(a)$,

(7) $a \models \exists x\alpha(x) \Leftrightarrow \exists A \subset\subset K\big(\bigcap A \leqslant a$ and $\forall b \in A\, \exists u \in U(b)\big(b \models \alpha(u)\big)\big)$.

Here, we mean by $A \subset\subset K$ that A is a finite subset of K.

Next, we will prove the soundness for this semantics, but the soundness theorem does not hold unconditionally. Before we express the condition, we prove the weak soundness which holds unconditionally.

Let Γ and γ be a set of formulas and a formula, respectively. An expression of the form $\Gamma \to \gamma$ or $\Gamma \to$ is called a *sequent*. A sequent $\{\alpha_1, \ldots, \alpha_n\} \cup \Gamma \to \gamma$ is denoted by $\alpha_1, \ldots, \alpha_n, \Gamma \to \gamma$. If all formulas appearing in a given sequent are closed formulas of $L(\mathbf{M})$, the sequent is called a *closed sequent of* **M**. A closed sequent $\Gamma \to \gamma$ (or $\Gamma \to$) is *valid in Kripke model* $\langle \mathbf{M}; \models \rangle$, if for any $a \in K$ $\forall \alpha \in \Gamma(a \models \alpha)$ and $\gamma \in L(a)$ imply $a \models \gamma$ (or for any $a \in K$ there exists $\alpha \in \Gamma$ such that a *non* $\models \alpha$, respectively). A sequent is *valid in* $\langle \mathbf{M}; \models \rangle$, if any closed sequent obtained from it by substituting constants of $L(\mathbf{M})$ for free variables is valid in $\langle \mathbf{M}; \models \rangle$. A rule of inference is *valid in* $\langle \mathbf{M}; \models \rangle$, if the lower sequent of it is valid in $\langle \mathbf{M}; \models \rangle$ whenever all the upper sequent(s) of it are valid in $\langle \mathbf{M}; \models \rangle$.

It is shown below that every initial sequent and most of inference rules of Gentzen's LJ are valid in every Kripke model.

LEMMA 1.4. *For any* $\alpha \in W(\mathbf{M})$ *and* $a, b, c \in K$,

(1) $a \models \alpha \Rightarrow \alpha \in L(a)$,

(2) $a = \infty(a)$ *and* $\alpha \in L(a) \Rightarrow a \models \alpha$,

(3) $a \models \alpha$ *and* $b \models \alpha$ *and* $a \cap b \leqslant c \Rightarrow c \models \alpha$.

PROOF. All are shown by induction on the length of α. We will show only (3), since the proofs of (1) and (2) are easy. We first remark that $\alpha \in L(c)$ by (1) and Definition 1.2(3). We have six cases, depending on the outermost logical symbol of α.

(i) The case where α is of the form $\beta \supset \gamma$. Suppose that $c \leqslant d \in K$ and $d \models \beta$. By $a \cap b \leqslant d$, Definitions 1.1(3) and 1.2(1), there exists $a' \in K$ such that $a' \cap b \leqslant d$, $a' \geqslant a$ and $a' \geqslant d \cap \infty(a \cap b)$. By using Definition 1.2(1) once more, there exists $e \in K$ such that $a' \geqslant d \cap e$ and $e = \infty(e) \geqslant \infty(a \cap b)$. By $\beta \in L(e)$ and the above Lemma 1.4(2), we have $e \models \beta$. Then, by $d \models \beta$ and the hypothesis of induction, we have also $a' \models \beta$. By $a' \geqslant a$, $a \models \beta \supset \gamma$ and $a' \models \beta$, we have $a' \models \gamma$. Similarly, there exists $b' \in K$ such that $a' \cap b' \leqslant d$ and $b' \models \gamma$. Hence, by the hypothesis of induction, we get $d \models \gamma$.

(ii) The case where α is of the form $\beta \vee \gamma$. By $a \models \beta \vee \gamma$, there exist $a_1, a_2 \in K$ such that $a_1 \cap a_2 \leqslant a$, $a_1 \models \beta$, $a_2 \models \gamma$, $\beta \in L(a_2)$ and $\gamma \in L(a_1)$. By $b \models \beta \vee \gamma$, there exist $b_1, b_2 \in K$ such that $b_1 \cap b_2 \leqslant b$, $b_1 \models \beta$, $b_2 \models \gamma$, $\beta \in L(b_2)$ and $\gamma \in L(b_1)$. By $c \geqslant a \cap b \geqslant (a_1 \cap b_1) \cap (a_2 \cap b_2)$ and Definition 1.2(1), there exist $d, e \in K$ such that $c \geqslant d \cap e$, $d \geqslant a_1 \cap b_1$ and $e \geqslant a_2 \cap b_2$. By the hypothesis of induction, $d \models \beta$ and $e \models \gamma$. Hence, by $\beta \in L(e)$ and $\gamma \in L(d)$, we have $c \models \beta \vee \gamma$.

(iii) The case where α is of the form $\beta \wedge \gamma$. Easy.

(iv) The case where α is of the form $\neg \beta$. Similar to (i).

(v) The case where α is of the form $\forall x \beta(x)$. Suppose that $c \cap e \leqslant d \in K$, $e = \infty(e) \in K$, $u \in U(d)$ and $\forall x \beta(x) \in L(e)$. By $d \geqslant c \cap e \geqslant (a \cap \infty(b \cap e)) \cap \cap (b \cap \infty(a \cap e))$, $\infty(a \cap \infty(b \cap e)) = \infty(b \cap \infty(a \cap e))$ and Definition 1.2 (4), $a \cap \infty(b \cap e)$ and $b \cap \infty(a \cap e)$ have extensions a', b' in K, resp., such that $d \geqslant a' \cap b'$, $U(a') \cap U(b') = U(d)$. By using again Definition 1.2(1), there exists $a'' \in K$ such that $a' \geqslant a \cap a''$ and $a'' \geqslant \infty(b \cap e)$. By $a \models \forall x \beta(x)$, $u \in U(a')$ and $\forall x \beta(x) \in L(a'')$, we have $a' \models \beta(u)$. Similarly, $b' \models \beta(u)$. Hence, by the hypothesis of induction, we have $d \models \beta(u)$.

(vi) The case where α is of the form $\exists x \beta(x)$. This is the only case which can be proved without the hypothesis of induction. Q.E.D.

The following lemma will simplify the proof of the soundness theorem. By the lemma, when we extend a valuation on a to $W(\mathbf{M})$, it suffices for us to look at elements of K whose universes include $U(a)$.

LEMMA 1.5. *$a \models \alpha \vee \beta \Leftrightarrow \exists b, c \in K (b \cap c \leqslant a$ and $b \models \alpha$ and $c \models \beta$ and $U(b) \cap U(c) = U(a))$ and $\alpha, \beta \in L(a)$.*

PROOF. The $\Leftarrow$ part is trvial. Suppose that $a \models \alpha \vee \beta$.
Then, there exist $b', c' \in K$ such that $b' \cap c' \leqslant a$, $b' \models \alpha$, $c' \models \beta$, $\alpha \in L(c')$ and $\beta \in L(b')$. By Definition 1.2(4), b' and c' have extensions b and c in K, resp., that $b \cap c \leqslant a$ and $U(b) \cap U(c) = U(a)$. By Lemma 1.4(2) (3), $b \models \alpha$ and $c \models \beta$. Q.E.D.

The following lemma can be proved similarly to Lemma 2.2 in Chapter 5 of Fitting [1].

LEMMA 1.6. *If* $\alpha(u), \Gamma \to \gamma$ (*or* $\Gamma \to \alpha(u)$) *is not valid in some Kripke model and if* v *is a constant which does not occur in* Γ, *in* γ *or in* $\alpha(u)$, *then there exists a Kripke model such that* $\alpha(v), \Gamma \to \gamma$ (or $\Gamma \to \alpha(v)$, *respectively*) *is not valid in it.*

LEMMA 1.7. (WEAK SOUNDNESS THEOREM). *Axioms and rules of inference except* $(\to\forall)$ *and* $(\exists\to)$ *of Gentzen's LJ are valid in every Kripke model.*

PROOF. It is trivial as for axioms. We will prove that (some) one of the upper sequent(s) of a given rule of inference is not valid in some Kripke model if the lower sequent is not valid in some Kripke model. In the following, we will show this only for six cases where the rules of inference are $(\vee\to)$, $(\to\vee)$, $(\supset\to)$, $(\to\supset)$, $\forall\to)$ and $(\to\exists)$.

(i) $(\vee\to)$. Suppose that $\alpha\vee\beta, \Gamma\to\delta$ is not valid in some Kripke model $\langle \mathbf{M}, \models\rangle$ ($\mathbf{M} = \langle M; \infty, \cap, K, U\rangle$). Then, there exists $a \in K$ such that $a \models \alpha\vee\beta$, $\forall\gamma \in \Gamma\,(a \models \gamma)$, a *non* $\models \delta$ and $\delta \in L(a)$. By $a \models \alpha\vee\beta$ and Lemma 1.5, there exist $b, c \in K$ such that $b \cap c \leqslant a$, $b \models \alpha$, $c \models \beta$, $U(b) \supset U(a)$ and $U(c) \supset U(a)$. By Definitions 1.1(3) and 1.2(1), there exist $b', c' \in K$ such that $b' \cap c' \leqslant a$, $b' \geqslant a \cap \infty(b\cap c)$, $b' \geqslant b$, $c' \geqslant a \cap \infty(b \cap c)$ and $c' \geqslant c$. By a *non* $\models \delta$ and Lemma 1.4(3), b' *non* $\models \delta$ or c' *non* $\models \delta$. We get $\delta \in L(b')$ and $\delta \in L(c')$. Consider the case when b' *non* $\models \delta$. By $b' \geqslant a \cap \infty(b\cap c)$ and Definition 1.2(1), there exists $d \in K$ such that $b' \geqslant a \cap d$ and $d \geqslant \infty(b \cap c)$. By $d = \infty(d)$ and $U(d) \supset U(a)$, we have $\forall\gamma \in \Gamma(d \models \gamma)$. Further, by $b' \geqslant b$, $b' \models \alpha$. Hence, $\alpha, \Gamma \to \delta$ is not valid in $\langle \mathbf{M}, \models\rangle$. In the case of c' *non* $\models \delta$, we have similarly that $\beta, \Gamma \to \delta$ is not valid in $\langle \mathbf{M}, \models\rangle$.

(ii) $(\to\vee)$. Suppose that $\Gamma \to \alpha\vee\beta$ is not valid in some Kripke model $\langle \mathbf{M}, \models\rangle$. Then, there exists $a \in K$ such that $\forall\gamma \in \Gamma(a \models \gamma)$, a *non* $\models \alpha\vee\beta$ and $\alpha, \beta \in L(a)$. By $a \cap \infty(a) \leqslant a \in K$ and Definition 1.2(1), there exists $b \in K$ such that $a \cap b \leqslant a$ and $b \geqslant \infty(a)$. By $b \models \beta$ (by $\beta \in L(b)$), we have $a \models \alpha\vee\beta$ if $a \models \alpha$. This contradicts a *non* $\models \alpha\vee\beta$. Hence, a *non* $\models \alpha$. Therefore $\Gamma \to \alpha$ is not valid in $\langle \mathbf{M}, \models\rangle$.

(iii) $(\supset\to)$. Suppose that $\alpha \supset \beta, \Gamma \to \delta$ is not valid in some Kripke model $\langle \mathbf{M}, \models\rangle$. Then, there exists $a \in K$ such that $a \models \alpha \supset \beta$, $\forall\gamma \in \Gamma(a \models \gamma)$ and a *non* $\models \delta$. By $a \models \alpha \supset \beta$, we have a *non* $\models \alpha$ or $a \models \beta$. $\Gamma \to \alpha$ is not valid in $\langle \mathbf{M}, \models\rangle$ if a *non* $\models \alpha$. $\beta, \Gamma \to \delta$ is not valid in $\langle \mathbf{M}, \models\rangle$ if $a \models \beta$.

(iv) $(\to\supset)$. Suppose that $\Gamma \to \alpha \supset \beta$ is not valid in some $\langle \mathbf{M}, \models\rangle$. Then, there exists $a \in K$ such that $\forall\gamma \in \Gamma(a \models \gamma)$, a *non* $\models \alpha \supset \beta$ and $\alpha, \beta \in L(a)$. By a *non* $\models \alpha \supset \beta$, there exists $b \in K$ such that $b \geqslant a$, $b \models \alpha$ and b *non* $\models \beta$. By $b \geqslant a$ and Lemma 1.4(3), $b \models \gamma$ for any $\gamma \in \Gamma$. Since it is obvious that $\beta \in L(b)$, $\alpha, \Gamma \to \beta$ is not valid in $\langle \mathbf{M}, \models\rangle$.

(v) $(\forall\to)$. Suppose that $\forall x\alpha(x), \Gamma \to \delta$ is not valid in some $\langle \mathbf{M}, \models\rangle$. Then, there exists $a \in K$ such that $a \models \forall x\alpha(x)$, $\forall\gamma \in \Gamma(a \models \gamma)$, a *non* $\models \delta$ and $\delta \in L(a)$. By $a \models \forall x\alpha(x)$, $a \models \alpha(u)$ for any $u \in U(a)$. Hence, by $u \in U(a)$,

$\alpha(u), \Gamma \to \delta$ is not valid in $\langle \mathbf{M}, \models \rangle$ if u occurs in $\forall x\alpha(x)$, in Γ, or in δ. Otherwise, by Lemma 1.6, $\alpha(u), \Gamma \to \delta$ is not valid in some model.

(vi) $(\to\exists)$. Suppose that $\Gamma \to \exists x\alpha(x)$ is not valid in some $\langle \mathbf{M}, \models \rangle$. Then, there exists $a \in K$ such that $\forall \gamma \in \Gamma (a \models \gamma)$, a *non* $\models \exists x\alpha(x)$ and $\exists x\alpha(x) \in L(a)$. By a *non* $\models \exists x\alpha(x)$, a *non* $\models \alpha(u)$ for any $u \in U(a)$. Hence, $\Gamma \to \alpha(u)$ is not valid in $\langle \mathbf{M}, \models \rangle$ if u occurs in Γ or in $\exists x\alpha(x)$. Otherwise, by Lemma 1.6. Q E D

Unconditionally, the rules of inference $(\to\forall)$ and $(\exists\to)$ are not valid. They are valid in normal models defined below.

DEFINITION 1.8. A Kripke model $\langle \mathbf{M}, \models \rangle$ $(\mathbf{M} = \langle M; \infty, \cap, K, U \rangle)$ is *normal* (or a valuation $\models$ of $\mathbf{M}$ is *normal*), if the following two conditions are satisfied for any $a, b, c \in K$ such that $a \cap c \leqslant b$ and $c = \infty(c)$, for any $u \in U(b)$, and for any closed formula $\forall x\alpha(x) \in L(c)$:

(1) $\forall d \in K \forall v \in U(d)\big(a \leqslant d \Rightarrow d \models \alpha(v)\big) \Rightarrow b \models \alpha(u)$,

(2) $a \models \alpha(w) \Rightarrow \exists a' \in K \exists v \in U(a')\big(a' \geqslant a \cap c,\ a' \models \alpha(v),$
$U(a') \supset U(b)$ and $a' \cap \infty(b) \leqslant b\big)$.

REMARK. A model $\langle \mathbf{M}, \models \rangle$ is always normal if the set $\{\infty(b) | b \in M\}$ is a singleton.

For normal models, a lemma corresponding to Lemma 1.5 holds also for logical symbols $\forall$ and $\exists$.

LEMMA 1.9. *If a Kripke model is normal, then for any* $a \in K$, *the following holds*:

(1) $a \models \forall x\alpha(x) \Leftrightarrow \forall b \in K \forall u \in U(b)\big(a \leqslant b \Rightarrow b \models \alpha(u)\big)$,

(2) $a \models \exists x\alpha(x) \Leftrightarrow \exists A \subset\subset K\Big[\bigcap A \leqslant a$ *and* $\forall b \in A\Big(U(a) \subset U(b)$ and
$\exists u \in U(b)\big(b \models \alpha(u)\big)\Big)\Big]$.

PROOF. (1). The $\Rightarrow$ part is trivial (it holds without the condition that the model is normal). Suppose (*) $\forall b \in K\ \forall u \in U(b)\big(a \leqslant b \Rightarrow b \models \alpha(u)\big)$. At first, we have $\forall x\alpha(x) \in L(a)$ since $a \models \alpha(u)$ by (*) (substitute a for b). Next, suppose that $b \geqslant a \cap c$, $c = \infty(c)$, $\forall x\alpha(x) \in L(c)$ and $u \in U(b)$. Then, by Definition 1.8(1), we have $b \models \alpha(u)$.

(2). The $\Leftarrow$ part is trivial. We will show the $\Rightarrow$ part. By $a \models \exists x\alpha(x)$, there exist $\{a'_i | i \in I\} \subset\subset K$ and $\{u'_i | i \in I\}$ such that $\bigcap_{i \in I} a'_i \leqslant a$ and $a'_i \models \alpha(u'_i)$ for any $i \in I$. By Definition 1.2(1) and (4), there exists a set $\{a''_i | i \in I\} \subset\subset K$ such that $a''_i \geqslant a'_i \cap \infty(\bigcap a'_i)$, $U(a''_i) \supset U(a)$ and $\bigcap a''_i \leqslant a$ for each $i \in I$. By $a'_i \models \alpha(u'_i)$ and Definition 1.8(2), there exist $a_i \in K$ and $u_i \in U(a_i)$ such that $a_i \geqslant a'_i \cap \infty(\bigcap a'_i)$, $a_i \models \alpha(u_i)$, $U(a_i) \supset U(a''_i)$ and $a_i \cap \infty(a''_i) \leqslant a''_i$. By $a_i \cap \infty(a''_i) \leqslant a''_i \in K$ and Definition 1.2(1), there exists $b_i \in K$ such that $a_i \cap b_i \leqslant a''_i$ and $b_i \geqslant \infty(a''_i)$. Let A be the set $\{a_i | i \in I\} \cup \{b_i | i \in I\}$. Then, we have $\bigcap A = \bigcap_{i \in I}(a_i \cap b_i) \leqslant \bigcap_{i \in I} a''_i \leqslant a$, $\forall c \in A\big(U(a) \subset U(c)\big)$ and $\forall c \in A \exists u \in U(c)\ \big(c \models \alpha(u)\big)$, since $U(a) \subset U(a_i)$, $U(a) \subset U(b_i)$, $a_i \models \alpha(u_i)$ and $\forall w \in U(b_i)\big(b_i \models \alpha(w)\big)$. Q.E.D.

THEOREM 1.10 (SOUNDNESS THEOREM). *If a sequent $\Gamma \to \Delta$ is provable in LJ, then $\Gamma \to \Delta$ is valid in every normal Kripke model.*

PROOF. By weak soundness theorem, it suffices to show that the rules of inference ($\to\forall$) and ($\exists\to$) are valid in every normal model.

($\to\forall$). Suppose that $\Gamma \to \forall x a(x)$ is not valid in some normal Kripke model $\langle \mathbf{M}, \models \rangle$ ($\mathbf{M} = \langle M; \infty, \cap, K, U \rangle$). Then, there exists $a \in K$ such that $a \models \gamma$ for any $\gamma \in \Gamma$, a *non* $\models \forall x a(x)$ and $\forall x a(x) \in L(a)$. By a *non* $\models \forall x a(x)$ and Lemma 1.9(1), there exist $b \in K$ and $u \in U(b)$ such that $a \leqslant b$ and b *non* $\models a(u)$. Since $b \models \gamma$ for any $\gamma \in \Gamma$ by $a \leqslant b$, $\Gamma \to a(x)$ is not valid in $\langle \mathbf{M}, \models \rangle$.

($\exists\to$). Suppose that $\exists x a(x), \Gamma \to \delta$ is not valid in some normal Kripke model $\langle \mathbf{M}, \models \rangle$ ($M = \langle M; \infty, \cap, K, U \rangle$). Then, there exists $a \in K$ such that $a \models \exists x a(x)$, $a \models \gamma$ for any $\gamma \in \Gamma$, a *non* $\models \delta$ and $\delta \in L(a)$. By $a \models \exists x a(x)$ and Lemma 1.9(2), there exist $\{a_i \mid i \in I\} \subset\subset K$ and $\{u_i \mid i \in I\}$ such that $\bigcap a_i \leqslant a$, $U(a_i) \supset U(a)$ and $a_i \models a(u_i)$ for any $i \in I$. By Definitions 1.1(3) and 1.2(1), there exists $\{a'_i \mid i \in I\} \subset\subset K$ such that $a'_i \geqslant a_i$, $\bigcap a'_i \leqslant a$ and $a'_i \geqslant a \cap \infty(\bigcap a_i)$. By $\bigcap a_i \leqslant a$ and a *non* $\models \delta$, there exists $k \in I$ such that a'_k *non* $\models \delta$. By $a'_k \geqslant a \cap \infty(\bigcap a_i)$ and $U(a) \subset U(a_i)$ for any $i \in I$, we have $a'_k \models \gamma$ for any $\gamma \in \Gamma$. Further, $a'_k \models a(u_k)$ and $\delta \in L(a'_k)$. Hence, $a(x), \Gamma \to \delta$ is not valid in $\langle \mathbf{M}, \models \rangle$. Q.E.D.

2. The relation between new semantics and old semantics, and Completeness Theorem

In the following, we will call a Kripke's original frame an *O-Kripke frame*. And we will call a new Kripke frame simply a Kripke frame. For an O-Kripke frame $\langle K, U \rangle$ (where K is a partially ordered set and U is a mapping from K to a family of sets), $L(K, U)$ denotes the set of all formulas valid in $\langle K, U, \models \rangle$ for any valuation $\models$ of $\langle K, U \rangle$. For a Kripke frame $\mathbf{M}$, $L(\mathbf{M})$ denotes also the set of all formulas valid in $\langle \mathbf{M}, \models \rangle$ for any normal valuation $\models$ of $\mathbf{M}$. Both $L(K, U)$ and $L(\mathbf{M})$ are logics; that is, sets of formulas are closed under modus ponens, the generalization and the substitution (cf. [5]). To show that $L(\mathbf{M})$ is a logic, we must use Lemmas 1.5 and 1.9.

Let us suppose we have an O-Kripke frame $\langle K, U \rangle$. We will construct a Kripke frame $\mathbf{M}$ such that $L(K, U) = L(\mathbf{M})$. Then, using the construction, we will prove the completeness theorem.

A subset A of K is said to be *open* if, for any $x \in A$ and any $y \in K$, $y \leqslant x$ implies $y \in A$. Define $O(K)$ to be the set of all open subsets of K. For any $A, B \in O(K)$, their intersection $A \cap B$ also belongs to $O(K)$. Let ∞ be the mapping from $O(K)$ to $O(K)$ such that $\infty(A) = K$ for any $A \in O(K)$. Further, for any $A, B \in O(K)$, their union $A \cup B$ also belongs to $O(K)$. Hence, if $A \cap B \subset C$ for $A, B, C \in O(K)$ then there exists $A' \in O(K)$ such that $A' \cap B \subset C$, $A' \supset A$ and $A' \supset C \cap K$, taking

$A \cup C$ for A'. Therefore, the triple $\langle O(K); \infty, \cap \rangle$ is a ∞-distributive semilattice. For any $x \in K$, let $h(x)$ be the set $\{y \in K \mid x \not\leqslant y\}$. Then, $h(x) \in O(K)$ for any $x \in K$. We define a subset K^* of $O(K)$ by $K^* = \{h(x) \mid x \in K\} \cup \{K\}$.

LEMMA 2.1. *Let A_i belong to $O(K)$ (for every $i \in I$) and B belong to K^*. If $\bigcap_{I \ni i} A_i \subset B$, then there exists $i \in I$ such that $A_i \subset B$.*

PROOF. In the case of $B = K$, it is obvious. Let $B \neq K$. Then, there exists $b \in K$ such that $B = h(b) = \{x \in K \mid b \not\leqslant x\}$. Clearly, the set B is the maximum element in elements of $O(K)$ not containing b. By $\bigcap A_i \subset \subset \{x \in K \mid b \not\leqslant x\}$, we have $b \notin \bigcap A_i$. Hence, there exists $i \in I$ such that $b \notin A_i$. Therefore, we have $A_i \subset B$. Q.E.D.

By this lemma, we can easily show that K^* is a frame subset of $\langle O(K); \infty, \cap \rangle$. Define a mapping U^* from K^* to the power set $\bigcup_{x \in K} U(x)$ by $U^*(h(x)) = U(x)$ and $U^*(K) = \bigcup_{x \in K} U(x)$. Then, the mapping U^* is an universe function of $\langle O(K); \infty, \cap, K^* \rangle$. Let $\mathbf{M}$ be the structure $\langle O(K); \infty, \cap, K^*, U^* \rangle$. From Remark in the previous section, it follows that any valuation of $\mathbf{M}$ is normal.

LEMMA 2.2. *For any valuation $\vDash$ of $\mathbf{M}$ and any $a \in K^*$:*
(1) $a \vDash \alpha \vee \beta \Leftrightarrow (a \vDash \alpha$ or $a \vDash \beta)$ and $\alpha, \beta \in L(a)$,
(2) $a \vDash \exists x \alpha(x) \Leftrightarrow \exists u \in U^*(a)(a \vDash \alpha(u))$.

PROOF. (1). The $\Rightarrow$ part is trivial. Suppose $a \vDash \alpha \vee \beta$. First, by Lemma 1.4(1), we have $\alpha, \beta \in L(a)$. There exist $b, c \in K^*$ such that $b \cap c \leqslant a$, $b \vDash \alpha$ and $c \vDash \beta$. By Lemma 2.1, $b \leqslant a$ or $c \leqslant a$. Hence, $a \vDash \alpha$ or $a \vDash \beta$.

(2). The $\Leftarrow$ part is trivial. Suppose $a \vDash \exists x \alpha(x)$. Then, there exists $A^* \subset \subset K^*$ such that $\bigcap A^* \leqslant a$ and $\exists u \in U^*(b)(b \vDash \alpha(u))$ for any $b \in A^*$. By Lemma 2.1, there exists $b \in A^*$ such that $b \leqslant a$. For some $u \in b$, $b \vDash \alpha(u)$. Thus, $u \in U^*(a)$ and $a \vDash \alpha(u)$. Q.E.D.

Both lemmas presented below can be proved by induction on the length of formulas. Moreover, they can be obviously proved since a valuation of a Kripke frame coincides with a valuation of an O-Kripke frame by Lemmas 1.9(1) and 2.2.

LEMMA 2.3. *For a valuation $\vDash$ of an O-Kripke frame $\langle K, U \rangle$, we define a subset $\vDash^*$ of $K^* \times W(\mathbf{M})$ as follows:*
(1) $h(a) \vDash^* \alpha \Leftrightarrow a \vDash \alpha$, for any $\alpha \in W(\mathbf{M})$ and any $a \in K$,
(2) $K \vDash^* \alpha$ for any $\alpha \in W(\mathbf{M})$.
Then, the subset $\vDash^*$ is a valuation of $\mathbf{M}$.

LEMMA 2.4. *For a valuation $\vDash^*$ of $\mathbf{M}$, we define a subset $\vDash$ of $K \times W(\mathbf{M})$ as follows: $a \vDash \alpha \Leftrightarrow h(a) \vDash^* \alpha$ for any $\alpha \in W(\mathbf{M})$ and any $a \in K$. Then, the subset $\vDash$ is a valuation of $\langle K, U \rangle$.*

By two lemmas given above, we have

THEOREM 2.5. $L(K, U) = L(\mathbf{M})$.

By using Lemma 2.3, we will show the completeness theorem for new semantics.

THEOREM 2.6. *The following three conditions are equivalent:*
(1) $\Gamma \to \Delta$ *is provable in* LJ,
(2) $\Gamma \to \Delta$ *is valid in every normal Kripke model,*
(3) $\Gamma \to \Delta$ *is valid in every O-Kripke model.*

PROOF. From (1), (2) follows by Theorem 1.10. By Kripke's original completeness theorem (using Henkin's construction), (1) follows from (3). So, it remains to show that (3) follows from (2). Suppose that $\Gamma \to \Delta$ is not valid in some O-Kripke model $\langle K, U, \models \rangle$. Then, by Lemma 2.3, $\Gamma \to \Delta$ is not valid in $\langle \mathbf{M}, \models^* \rangle$. Q.E.D.

CORRECTIONS TO [2]. Mr. Shin-ichi Yokota has informed the author of the following corrections to [2]. In Theorem 2.6, replace "$\neg q \subset \mathrm{LK} \cap$" by "$\neg q = \mathrm{LK} \cap$". In Corollary 2.7, delete "Both" and "and $\mathrm{LJ} + \exists x p(x) \supset \supset \forall x p(x) \vee q \vee \neg q$".

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