SACHIO HIROKAWA
YUICHI KOMORI
IZUMI TAKEUTI

A Reduction Rule for Peirce Formula*

Abstract. A reduction rule is introduced as a transformation of proof figures in implicational classical logic. Proof figures are represented as typed terms in a λ-calculus with a new constant $P^{((\alpha\to\beta)\to\alpha)\to\alpha}$. It is shown that all terms with the same type are equivalent with respect to β-reduction augmented by this P-reduction rule. Hence all the proofs of the same implicational formula are equivalent. It is also shown that strong normalization fails for βP-reduction. Weak normalization is shown for βP-reduction with another reduction rule which simplifies α of $((\alpha \to \beta) \to \alpha) \to \alpha$ into an atomic type.

Key words: Curry-Howard isomorphism, typed lambda-calculus, classical logic.

1991 Mathematics Subject Classification: 03F05, 68Q43.

1. A combinatory reduction rule for the Peirce formula

Implicational theorems in classical logic are deduced from axiom schemes

$$
\begin{aligned}
S &: (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \\
K &: \alpha \to \beta \to \alpha \\
P &: ((\alpha \to \beta) \to \alpha) \to \alpha
\end{aligned}
$$

by substitution and modus ponens. The last formula is known as the *Peirce formula.* The first two determine intuitionistic logic and have the reduction rules $Sxyz \to xz(yz)$ and $Kxy \to x$. These two reduction rules correspond to the normalization of proof figures in the Natural Deduction System [4], or equivalently to the β-reduction of typed λ-terms [1, 5]. What is a reduction rule for the Peirce formula? What kind of equivalence of proofs is derived from such a reduction rule? We found a reduction rule and show that all the proofs of the same formula are convertible with respect to β-reduction and this reduction [1].

*This work was partially supported by a Grant-in-Aid for General Scientific Research No. 05680276 of the Ministry of Education, Science and Culture, Japan and by Japan Society for the Promotion of Science.

[1]The material in the section 2 was presented to the Logic Colloquium '94; see [2] for the abstract.

Presented by **Hiroakira Ono**; *Received* May 27, 1994; *Revised* May 31, 1995

Studia Logica **56**: 419–426, 1996.

We use lower case Roman letters $a, b, c, \cdots$ for type variables. The lower case Greek letters $\alpha, \beta, \gamma, \cdots$ are used for general types. We formalize the system as a Natural Deduction System with the Peirce formula as an axiom scheme. The proof figures in this system are represented by typed λ-terms with a new constant $P^{((\alpha\to\beta)\to\alpha)\to\alpha}$. We call such terms typed λP-terms. We omit type information when it is clear from the context. For example, we write

$$\lambda y^{\alpha\to\beta}.x^{\alpha\to\beta\to\gamma}z^{\alpha}(yz)$$

as an abbreviation of

$$(\lambda y^{\alpha\to\beta}.((x^{\alpha\to\beta\to\gamma}z^{\alpha})^{\beta\to\gamma}(y^{\alpha\to\beta}z^{\alpha})^{\beta})^{\gamma})^{(\alpha\to\beta)\to\gamma}.$$

We think that the following reduction rule (P-reduction) is an answer to our question.

DEFINITION 1.1 (P-reduction)

$$x^{\alpha\to\beta}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha}) \xrightarrow{P} x^{\alpha\to\beta}(y^{(\alpha\to\beta)\to\alpha}x^{\alpha\to\beta})$$

This reduction represents the following transformation of proof figures which eliminates superfluous occurrence of the Peirce axiom.

$$\dfrac{\begin{matrix}\vdots\\ M:\alpha\to\beta\end{matrix} \qquad \dfrac{P:((\alpha\to\beta)\to\alpha)\to\alpha \quad \begin{matrix}\vdots\\ N:(\alpha\to\beta)\to\alpha\end{matrix}}{PN:\alpha}}{M(PN):\beta}$$

$$\xrightarrow{P} \quad \dfrac{\begin{matrix}\vdots\\ M:\alpha\to\beta\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ N:(\alpha\to\beta)\to\alpha\end{matrix} \quad \begin{matrix}\vdots\\ M:\alpha\to\beta\end{matrix}}{NM:\alpha}}{M(NM):\beta}$$

Besides this interpretation of P-reduction, the reduction seems natural in the sense that it gives rise to the Peirce formula as follows. It is natural to require that the same variable has the same type in both sides of the reduction $x(Py) \to x(yx)$ and that the reduction preserves the type. Then the type of P is necessarily $((\alpha\to\beta)\to\alpha)\to\alpha$. In these two respects, we think that our reduction rule is a natural one.

2. Collapse of λP-terms of the same type

The P-reduction is a reduction rule for typed λP-terms not for type-free λP-terms. We cannot reduce $x^{\alpha\to\gamma}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha})$ unless $\gamma = \beta$. If we ignore this condition we have $K =_{\beta P} I$ (where $K = \lambda xy.x$ and $I = \lambda x.x$) by

$$I \overset{\beta}{\longleftarrow} I(II) \overset{P}{\longleftarrow} I(PI) \overset{\beta}{\longleftarrow} I(K(PI)I) \overset{P}{\longrightarrow} I(K(IK)I) \overset{\beta}{\longrightarrow} K.$$

Therefore all type-free λP-terms, hence all type-free λ-terms, are equivalent with respect to βP-convertibility. The two steps of P-reduction in this example are not allowed in the typed system, since the term PI has no type. Then we thought that these phenomena would not happen with respect to 'typed' P-reduction. And we raised the questions.

- Does 'typed' P-reduction, together with β-reduction, have the Church-Rosser property?
- Is βP-convertibility conservative with respect to β-convertibility?
- Do all the typed λP-terms collapse (i.e., are they all βP-convertible)?

The answer we obtained was an unexpected one as the following lemma shows that there is a λP-term which is reducible to the Church's numerals $\mathbf{1} = \lambda xy.xy$ and $\mathbf{2} = \lambda xy.x(xy)$. Both are in βP-normal form, so that βP-reduction does not have the Church-Rosser property. And βP-convertibility is not conservative with respect to β-convertibility. Moreover, Theorem 1 shows that all the λP-terms with the same type are βP-convertible.

LEMMA 2.1 $x^{\alpha\to\alpha}y^{\alpha} =_{\beta P} x(xy)$.

PROOF. We have the following two βP-reductions of the same typed λP-term

$$M^{\alpha} = (\lambda z^{\alpha}.x^{\alpha\to\alpha}(I^{\alpha\to\alpha}z))(P^{((\alpha\to\alpha)\to\alpha)\to\alpha}(\lambda v^{\alpha\to\alpha}.vy^{\alpha})),$$

where $I = \lambda u^{\alpha}.u$.

$$M \overset{P}{\longrightarrow} (\lambda z.x(Iz))((\lambda v.vy)(\lambda z.x(Iz))) \overset{\beta}{\longrightarrow} x(xy)$$

$$M \overset{\beta}{\longrightarrow} x(I(P(\lambda v.vy))) \overset{P}{\longrightarrow} x(I((\lambda v.vy)I)) \overset{\beta}{\longrightarrow} xy$$

Hence we have $x(xy) =_{\beta P} xy$. ∎

THEOREM 2.2 *$A^\alpha =_{\beta P} B^\alpha$ for any typed λP-term A^α and B^α with the same type α.*

PROOF. We define a pair by $[x_1, x_2] = \lambda d.dx_1x_2$. Then each projection is defined by $P_i = \lambda x_1x_2.x_i$ and satisfies $[x_1,x_2]P_i \xrightarrow{\beta} x_i$ $(i = 1,2)$. The exchange operator $X = \lambda z.[zP_2, zP_1]$ satisfies $X[x_1,x_2] \xrightarrow{\beta} [x_2,x_1]$. When both components have the same type α, the projection P_i has a type $\beta = \alpha \to \alpha \to \alpha$, a pair has type $\beta \to \alpha$ and the exchange operator X has type $(\beta \to \alpha) \to \beta \to \alpha$. Now we put $Z = \lambda nab.nX[a,b]P_1$. Then Z has a type $(((\beta \to \alpha) \to \beta \to \alpha) \to (\beta \to \alpha) \to \beta \to \alpha) \to \alpha \to \alpha \to \alpha$. By the definition of X we have the following reductions.

$$
\begin{array}{lll}
Z(\lambda xy.xy)AB & \xrightarrow{\beta} X[A,B]P_1 & \xrightarrow{\beta} [B,A]P_1 \xrightarrow{\beta} B \\
Z(\lambda xy.x(xy))AB & \xrightarrow{\beta} X(X[A,B])P_1 & \xrightarrow{\beta} [A,B]P_1 \xrightarrow{\beta} A
\end{array}
$$

By Lemma 2.1 we have $\lambda xy.xy =_{\beta P} \lambda xy.x(xy)$, hence $A^\alpha =_{\beta P} B^\alpha$. ■

A similar fact is known in category theory that states there is at most one morphism in $hom(A,B)$ for Boolean categories. See Proposition 8.3 and Historical comments on Section 8 of [3]. We need further work to obtain a clear explanation of this similarity.

In the proof of Lemma 2.1 and Theorem 2.2, it suffices to have P-reduction $x(Py) \to x(yx)$ only for the case that P is of type $((\alpha \to \alpha) \to \alpha) \to \alpha$. But we think that the meaning of P-reduction is in the case that P is of type $((\alpha \to \beta) \to \alpha) \to \alpha$ and $\alpha \neq \beta$. The type $((\alpha \to \alpha) \to \alpha) \to \alpha$ does not yield P-reduction. In fact, it is a type of the λ-term $\lambda x.x(\lambda z.x(\lambda y.y))$ and is provable in intuitionistic logic. So, even if we introduce a new constant Q which behaves like the λ-term with the reduction rule $Qx \to x(\lambda z.x(\lambda y.y))$, it does not cause the collapse.

3. Normalization of λP-terms

The P-reduction seems to be natural, but it does not enjoy strong normalization.

THEOREM 3.1 *Strong normalization fails for βP-reduction.*

PROOF. Let $M = (\lambda x^\alpha.P(\lambda y^{\alpha\to\alpha}.yz^\alpha))\,(P(\lambda y^{\alpha\to\alpha}.yz))$. Then we have the following reduction cycle.

$$\begin{array}{rcl} M & \xrightarrow{P} & (\lambda x.P(\lambda y.yz))((\lambda y.yz)(\lambda x.P(\lambda y.yz))) \\ & \xrightarrow{\beta} & (\lambda x.P(\lambda y.yz))((\lambda x.P(\lambda y.yz))z) \\ & \xrightarrow{\beta} & M \end{array}$$

Therefore strong normalization fails. ∎

The strategy of reducing the inner-most redex of highest degree fails. The degree of a type is the number of arrows in the type. The degree of a β-redex $(\lambda x^{\alpha}.M^{\beta})^{\alpha\to\beta}N^{\alpha}$ is the degree of $\alpha \to \beta$ and the degree of a P-redex $x^{\alpha\to\beta}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\ to\alpha})$ is the degree of $((\alpha \to \beta) \to \alpha) \to \alpha$. This strategy terminates for typed λ-terms. But the cycle in the proof of the theorem was created by this strategy. So this strategy does not terminate if we apply it to λP-terms.

The weak normalization holds if we add another reduction rule, which we call simplification of the Peirce formula. At the moment, we do not know whether weak normalization holds or not without simplification. A similar transformation is used in Prawitz [4] to restrict the elimination of double negation $(\alpha \to \bot) \to \bot \vdash \alpha$ to the case that α is atomic. Since our simplification does not create any overlap of redexes, the Church-Rosser theorem holds for β-reduction and the simplification. Therefore the convertibility induced by β-reduction and simplification gives a conservative extension to the β-convertibility.

DEFINITION 3.2 (Simplification of Peirce formula)

$$P^{(((\alpha_1\to\alpha_2)\to\beta)\to\alpha_1\to\alpha_2)\to\alpha_1\to\alpha_2} \to$$

$$\lambda x^{((\alpha_1\to\alpha_2)\to\beta)\to\alpha_1\to\alpha_2} y^{\alpha_1}.P^{((\alpha_2\to\beta)\to\alpha_2)\to\alpha_2}(\lambda z^{\alpha_2\to\beta}.x(\lambda u^{\alpha_1\to\alpha_2}.z(uy))y)$$

A one step of simplification decreases the degree of type of P. Therefore a consecutive simplifications terminates at a term in which every occurrence of P has a type of the form $((a \to \beta) \to a) \to a)$ with atomic type a.

DEFINITION 3.3 A Peirce formula $((\alpha \to \beta) \to \alpha) \to \alpha$ is *simple* iff α is atomic. A λP-term is *simplified* iff every occurrence of P has a simple Peirce formula.

DEFINITION 3.4 The grade $g(M)$ of a λP-term M is defined by

$$g(M) = \begin{cases} g(N) + 1 & if M = \lambda x.PN, \\ 0 & otherwise. \end{cases}$$

LEMMA 3.5 *If $Q^{(a\to\gamma)\to a}$ is a simplified λP-terms, then $g(Q) = g(Q[u := X])$ for any λP-term $X^{a\to\beta}$.*

PROOF. By induction on the structure of Q.
1 Q is atomic. Since Q has a type $(a \to \gamma) \to a$, it follows that $Q \neq P$ and $Q \neq u$. Therefore Q is a variable distinct from u. Hence we have $Q[u := X] = Q$ and Lemma holds.
2 $Q = MN$ is an application. Then $Q[u := X]$ is an application. Therefore the grades of Q and $Q[u := X]$ are zero.
3 $Q = \lambda x.R$ is an abstraction. Since Q has a type $(a \to \gamma) \to a$, R has an atomic type a. Hence R is not an abstraction. When u does not occur in Q as a free variable, $Q[u := X] = Q$ and therefore Lemma holds. So, we can suppose that $u \neq x$ and u appears free in R. Since u and R has types $a \to \beta$ and a, respectively, R cannot be atomic. Therefore, we can assume that $R = MN$.
3.1 $M \neq P$. Then,we have $g(Q) = 0$. On the other hand, we have $Q[u := X] = \lambda M[u := X]N[u := X]$. If $M[u := X] = P$ then $M = u$ and $X = P$, since $M \neq P$. But X cannot be equal to P, as X has a type $a \to \beta$. Thus $M[u := X] \neq P$. Hence $g(Q[u := X]) = 0$.
3.2 $R = PN$. Since $Q = \lambda x.PN$ is simplified, every occurrence of P has a simple Peirce formula. Hence the type of N has the form $(a \to \delta) \to a$ with some type δ. By the induction hypothesis for N, we have $g(N) = g(N[u := X])$. Thus we have $g(Q[u := X]) = g(\lambda x.PN[u := X]) = g(N[u := X]) + 1 = g(N) + 1 = g(Q)$. Hence Lemma holds. ■

LEMMA 3.6 *Let $M = X^{a\to\beta}(P^{((a\to\beta)\to a)\to a}Y^{(a\to\beta)\to a})$ be a simplified λP-term in β-normal form. If X and Y have no P-redex, then M is reducible to a simplified λP-term of the form XZ which has no β-redex and no P-redex.*

PROOF. By induction on $g(Y)$.
1 Base Step. $g(Y) = 0$.
1.1 Y is atomic. Since Y has a type $(a \to \beta) \to a$, it follows $Y \neq P$. Therefore $Y = y$ is a variable. Then we have $M = X(Py) \xrightarrow{P} X(yX)$. Since M is in β-normal form, X is not an abstraction. Therefore $X(yX)$ is in β and P-normal form.
1.2 $Y = QR$ is an application. Then we have $M = X(P(QR)) \xrightarrow{P} X(QRX)$, where $X(QRX)$ is in βP-normal form.
1.3 Y is an abstraction. Let $Y = \lambda u^{a\to\beta}.U^a$. Since U has an atomic type a, U is either a variable $y(\neq u)$ or an application.

1.3.1 $U = y(\neq u)$. Then we have a reduction $M = X(P(\lambda u.y)) \xrightarrow{P} X((\lambda u.y)X) \xrightarrow{\beta} Xy$. Therefore M is normalized to Xy.
1.3.2 $U = QR$. We have a reduction $M = X(P(\lambda u.QR)) \xrightarrow{P} X((\lambda u.QR)X) \xrightarrow{\beta} X(Q[u := X]R[u := X])$. Here firstly note that $Q \neq P$, since $g(Y) = 0$. Secondly note that $X \neq P$, since X has a type $a \to \beta$. Thirdly note that $X \neq PS$, since P has a simple Peirce formula. Hence $X(Q[u := X]R[u := X])$ is not a P-redex and no P-redex is created by the substitution $u := X$. Therefore the substitution does not create any β-redex. Thus $X(Q[u := X]R[u := X])$ is in βP-normal form.
2 Induction Step. $g(Y)>0$. Then we have $Y = \lambda u^{a\to\beta}.P^{((a\to\gamma)\to a)\to a)}Q^{(a\to\gamma)\to a}$ for some simplified λP-term Q and atomic type a. Consider the reduction $M = X(P(\lambda u.PQ)) \xrightarrow{P} X((\lambda u.PQ)X) \xrightarrow{\beta} X^{a\to\beta}(P^{((a\to\gamma)\to a)\to a}Q^{(a\to\gamma)\to a}[u := X])$.
2.1 $\beta \neq \gamma$. Then $X(PQ[u := X])$ is not a P-redex. Since the substitution does not create any redex, $X(PQ[u := X])$ is in βP-normal form.
2.2 $\beta = \gamma$. Then we have $g(Q) = g(Q[u := X])$ by Lemma 3.5. Therefore we have $g(Y) = g(\lambda u.PQ) = g(Q) + 1 = g(Q[u := X]) + 1 > g(Q[u := X])$. By the induction hypothesis for $Q[u := X]$, we can reduce $X(PQ[u := X])$ into a βP-normal form of the form XZ. ∎

THEOREM 3.7 (Weak Normalization) *Weak normalization theorem holds for the simplified λP-terms with respect to βP-reduction.*

PROOF. Firstly reduce the simplified λP-term into its β-normal form. As a result we obtain a simplified λP-term in which only P-redexes may remain. Secondly apply Lemma 3.6 repeatedly to the innermost P-redex. Each application of Lemma 3.6 does not create a new redex or a copy of P-redex. Hence it decreases the number of the P-redexes in the term. Eventually, we obtain a simplified λP-term which has no redex with respect to βP-reduction. ∎

Acknowledgment

Early study of this work took place when the first author was visiting Prof. M. Bunder at University of Wollongong. We thank him for his welcome encouragement. The authors express their thanks to Y. Akama, Prof. R. Hindley, E. Kyuma, H. Nakano and Prof. H. Ono for their inspiring discussions, and to the referee for valuable comments.

References

[1] Hindley, J. R and J. P. Seldin, 1986, *Introduction to Combinators and Lambda-Calculus,* Cambridge University Press, London.

[2] Hirokawa, S., Y. Komori and I. Takeuti, 1995, A reduction rule for the Peirce formula (abstract), *Bulletin of Symbolic Logic,* **1**, 239–240.

[3] Lambek, J. and P. J. Scott, 1986, *Introduction to higher order categorical logic,* Cambridge University Press, London.

[4] Prawitz, D., 1965, *Natural Deduction,* Almqvist and Wiksell, Stockholm.

[5] Prawitz, D., 1971, *Ideas and results in proof theory. In Proc. 2nd Scandinavian Logic Symposium,* ed. J.E. Fenstad. North-Holland, 235–307.

Sachio Hirokawa
Computer Science Laboratory
Faculty of Science
Kyushu University
Ropponmatsu 4-2-1
Fukuoka 810 JAPAN
hirokawa@rc.kyushu-u.ac.jp

Yuichi Komori
Department of Mathematics
Shizuoka University
Shizuoka 422 JAPAN
komori@sci.shizuoka.ac.jp

Izumi Takeuti
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-03 JAPAN
takeuti@math.metro-u.ac.jp