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A reduction rule for Peirce's formula makes all the terms with the same type equal

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A reduction rule for Peirce's formula makes all the terms with the same type equal *

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Abstract

A reduction rule is introduced by $x^{\alpha\to\beta}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha}) \xrightarrow{P} x(yx)$ as a transformation of proof figures in implicational classical logic. The proof figure are represented as typed λ-terms with a new constant $P^{((\alpha\to\beta)\to\alpha)\to\alpha}$. It is shown that all the terms with the same type are equivalent with respect to the β-reduction and this reduction. Hence all the proof of the same implicational formula are equivalent.

Implicational theorems in classical logic are constructed from axiom schemes $S : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$, $K : \alpha \to \beta \to \alpha$, $P : ((\alpha \to \beta) \to \alpha) \to \alpha$ by substitution and modus ponens. The last formula is known as *Peirce's formula.* The first two determine the intuitionistic logic and have the reduction rules $Sxyz \to xz(yz)$ and $Kxy \to x$. This two reduction rules correspond to the normalisation of proof figures in the Natural Deduction System [2], or equivalently to the β-reduction of typed λ-terms [1, 3]. What is a reduction rule for Peirce's formula? What kind of equivalence of proofs is derived from such reduction rule? We found a reduction rule and show that all the proofs of the same formula are convertible with respect to β-reduction and this reduction.

We formalise the system by the Natural Deduction System with Peirce's formula as an axiom scheme. The proof figures in this system are represented by typed λ-terms with a new constant $P^{((\alpha\to\beta)\to\alpha)\to\alpha}$. We call such

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terms typed λP-terms. We omit type information when it is clear from the context. For example, we write $\lambda y^{\alpha\to\beta}.x^{\alpha\to\beta\to\gamma}z^{\alpha}(yz)$ as an abbreviation of $(\lambda y^{\alpha\to\beta}.((x^{(\alpha\to\beta)\to\gamma}z^{\alpha})^{\beta\to\gamma}\,(y^{\alpha\to\beta}z^{\alpha})^{\beta})^{\gamma})^{(\alpha\to\beta)\to\gamma}$.

We think that the following reduction rule (P-reduction) is an answer to our question.

$$x^{\alpha\to\beta}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha}) \xrightarrow{P} x(yx)$$

The reduction represents the following transformation of proof figures which eliminates superfluous occurrence of Peirce's axiom.

$$\dfrac{\begin{array}{c}\vdots\\ M:\alpha\to\beta\end{array} \qquad \dfrac{P:((\alpha\to\beta)\to\alpha)\to\alpha \qquad \begin{array}{c}\vdots\\ N:(\alpha\to\beta)\to\alpha\end{array}}{PN:\alpha}}{M(PN):\beta}$$

$$\xrightarrow{P} \quad \dfrac{\begin{array}{c}\vdots\\ M:\alpha\to\beta\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ N:(\alpha\to\beta)\to\alpha\end{array} \qquad \begin{array}{c}\vdots\\ M:\alpha\to\beta\end{array}}{NM:\alpha}}{M(NM):\beta}$$

Besides this interpretation of P-reduction, the reduction seems natural in the sense that it gives rise to Peirce's formula as follows. It is natural to require that the same variable has the same type in both sides of the reduction $x(Py)\to x(yx)$ and that the reduction preserves the type, then the type of P is necessarily $((\alpha\to\beta)\to\alpha)\to\alpha$. In these two respects, we think that our reduction rule is a natural one.

The P-reduction is a reduction rule for typed λP-terms not for type-free λP-terms. We cannot reduce $x^{\alpha\to\gamma}(P^{((\alpha\to\beta)\to\alpha)\to\alpha}y^{(\alpha\to\beta)\to\alpha})$ unless $\gamma=\beta$. If we ignore this condition we have $K =_{\beta P} I$ ($K=\lambda xy.x$, $I=\lambda x.x$) by

$$I \overset{\beta}{\twoheadleftarrow} I(II) \overset{P}{\longleftarrow} I(PI) \overset{\beta}{\twoheadleftarrow} I(K(PI)I) \overset{P}{\longrightarrow} I(K(IK)I) \overset{\beta}{\twoheadrightarrow} K.$$

Therefore all type-free λP-terms, hence all type-free λ-terms, are equivalent with respect to βP-convertibility. The two steps of P-reduction in this example are not allowed, since the the term PI has no type. Then we thought that these phenomena would not happen with respect to 'typed' P-reduction. And we raised the questions.

- Does the 'typed' P-reduction, together with the β-reduction, have Church-Rosser property?
- Do all the typed λP-terms collapse (i.e., βP-convertible) or not?
- Is the βP-convertibility conservative to β-convertibility?

The answer we obtained was unexpected one as the following lemma shows that the Church's numerals $\mathbf{3} = \lambda xy.x(x(xy))$ and $\mathbf{4} = \lambda xy.x(x(x(xy)))$ are βP-convertible. Hence βP-convertibility is not conservative to β-convertibility. Moreover, the main theorem shows that all the λP-terms with the same type are βP-convertible.

Lemma $x^{\alpha\to\alpha}(x(xy^{\alpha})) =_{\beta P} x(x(x(xy)))$.

Proof. We have the following two βP-reductions of the same typed λP-term $M^{\alpha} = (\lambda z^{\alpha}.x^{\alpha\to\alpha}(xz))(P^{((\alpha\to\alpha)\to\alpha)\to\alpha}(\lambda u^{\alpha\to\alpha}.uy^{\alpha}))$.

$$M \xrightarrow{P} (\lambda z.x(xz))((\lambda u.uy)(\lambda z.x(xz))) \xrightarrow{\beta}\!\!\!\!\rightarrow x(x(x(xy)))$$
$$M \xrightarrow{\beta} x(x(P(\lambda u.uy))) \xrightarrow{P} x(x((\lambda u.uy)x)) \xrightarrow{\beta} x(x(xy))$$

Hence we have $x(x(x(xy))) =_{\beta P} x(x(xy))$. ∎

Theorem $A^{\alpha} =_{\beta P} B^{\alpha}$ for any typed λP-term A^{α} and B^{α} with the same type α.

Proof. We define a quintuple by $[x_1, x_2, x_3, x_4, x_5] = \lambda d.dx_1x_2x_3x_4x_5$. Then the projection of i-th component is defined by $p_i = \lambda x_1x_2x_3x_4x_5.x_i$ and satisfies $[x_1, x_2, x_3, x_4, x_5]p_i \xrightarrow{\beta}\!\!\!\!\rightarrow x_i$ $(i = 1, 2, 3, 4, 5)$. The shift-right operator $R = \lambda p.[pp_1, pp_1, pp_2, pp_3, pp_4]$ satisfies $R[x_1, x_2, x_3, x_4, x_5] \xrightarrow{\beta}\!\!\!\!\rightarrow [x_1, x_1, x_2, x_3, x_4]$. When every component has the same type α, a projection p_i has a type $\beta = \alpha \to \alpha \to \alpha \to \alpha \to \alpha \to \alpha$, a quintuple has $\beta \to \alpha$ and the shift-right operator R has $(\beta \to \alpha) \to \beta \to \alpha$. Now we put $X = \lambda nab.nR[a, b, b, b, b]p_5$. Then X has a type $(((\beta \to \alpha) \to \beta \to \alpha) \to (\beta \to \alpha) \to \beta \to \alpha) \to \alpha \to \alpha \to \alpha$. By definition of R we have the following reductions.

$$X(\lambda xy.x(x(xy)))AB \xrightarrow{\beta} (R(R(R[A, B, B, B, B])))p_5 \xrightarrow{\beta} [A, A, A, A, B]p_5 \xrightarrow{\beta} B$$
$$X(\lambda xy.x(x(x(xy))))AB \xrightarrow{\beta} (R(R(R(R[A, B, B, B, B]))))p_5 \xrightarrow{\beta} [A, A, A, A, A]p_5 \xrightarrow{\beta} A$$

By Lemma we have $\lambda xy.x(x(xy)) =_{\beta P} \lambda xy.x(x(x(xy)))$, hence $A^{\alpha} =_{\beta P} B^{\alpha}$. ∎

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