Game theory in economics and Blackwell determinacy from an intuitionistic point of view

Takako Nemoto

Outline of this talk

- von Neumann's game and the minimax theorem
- Blackwell game and its determinacy
- From an intuitionistic point of view?

von Neumann's zero sum game

I \ II	str.A	str.B
str.1	a_{1A}	a_{1B}
str.2	a_{2A}	a_{2B}

- Both player choose his strategy at the same time.

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- I wants to get as much as possible.
- II want to make his loss as little as possible.
- Is there an equilibrium point?

Example 1

- str.1 yields $1 at least

II / I	str.A	str.B
str.1	1	2
str.2	3	4

game 1

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- str.A's costs \$$3$ at most
- str.B's costs \$$4$ at most

$\longrightarrow$ II's value is 3

The optimal pair of strategies is (str.2, str.A).
The *value* of the game 3.

Example 2

- str.1 yields $\$1$ at least

I \ II	str.A	str.B
str.1	4	1
str.2	2	3

game 2

Example 2

- str.1 yields \$$1$ at least
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Example 2

- str.1 yields \$$1$ at least
- str.2 yields \$$2$ at least

$\rightarrow$ I's value is 2

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There is no optimal strategies!

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The existence of the equilibrium point

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the optimal strategies exists iff

$$\max_{i\in\{1,2\}} \min_{X\in\{A,B\}} a_{iX} = \min_{X\in\{A,B\}} \max_{i\in\{1,2\}} a_{iX}$$

Mixed strategy

Mixed strategy:

a probability distribution on the set of all strategies

	I \ II	str.A	str.B
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How about mixed strategy?

Minimax theorem

Theorem (von Neumann)
For any game, the pair of optimal mixed strategies exists, i.e.,

$$\max_{\sigma \in \mathrm{MS}_I} \min_{\tau \in \mathrm{MS}_{II}} \mathrm{E}(\sigma, \tau) = \min_{\tau \in \mathrm{MS}_{II}} \max_{\sigma \in \mathrm{MS}_I} \mathrm{E}(\sigma, \tau),$$

where

$\mathrm{E}(\sigma, \tau)$: the expected value of the game with I's mixed strategy σ and II's mixed strategy τ.

MS_I: the set of mixed strategies for player I

MS_{II}: the set of mixed strategies for player II

Blackwell games

"Infinite iteration" of von Neumann's game

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- Player II pays \$$f(\alpha, \beta)$ to player I.

Strategy and determinacy of Blackwell games

Let f be a given pay-off function.

Strategies: A function which assigns a probability distribution on X to every $\langle s, t\rangle \in X^{<\mathbb{N}} \times X^{<\mathbb{N}}$.

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I's value $E_I(f)$**:** $\sup_\sigma E_\sigma(f)$

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Blackwell game f is *determinate* if $E_I(f) = E_{II}(f)$.

Axiom of determinacy and Blackwell games

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- I wins iff $\alpha \in A$

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II		$\alpha(1)$		$\alpha(3)$		$\alpha(5)$		$\cdots$

- I wins iff $\alpha \in A$
- A is determinate if one of the player has a ws.

Theorem (Martin)

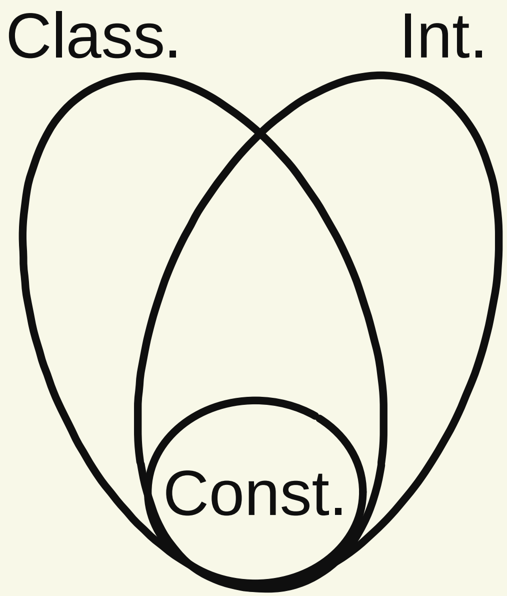
Axiom of determinacy $\rightarrow$

Determinacy of Blackwell game in $2^{\mathbb{N}}$

From an intuitionistic point of view

We work in “Brouwerian mathematics.”

- Logic is the intuitionistic logic.
- It has some mathematical axioms which is not included in the classical mathematics.



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Classical logic

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- $\varphi \vee \psi \leftrightarrow \neg(\neg \varphi \wedge \neg \psi)$ (de Morgan’s law)

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Axiom of intuitionistic mathematics 1

1st & 2nd axioms of continuous choice

For any relation $R \subseteq \mathcal{C} \times \mathbb{N}$ (resp. $\mathcal{C} \times \mathcal{C}$),
if, for any $\alpha \in \mathcal{C}$, there is β s.t. $R(\alpha, \beta)$,
then there is cont. f s.t., for all $\alpha \in \mathcal{C}$, $R(\alpha, f(\alpha))$.

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then there is cont. f s.t., for all $\alpha \in \mathcal{C}$, $R(\alpha, f(\alpha))$.

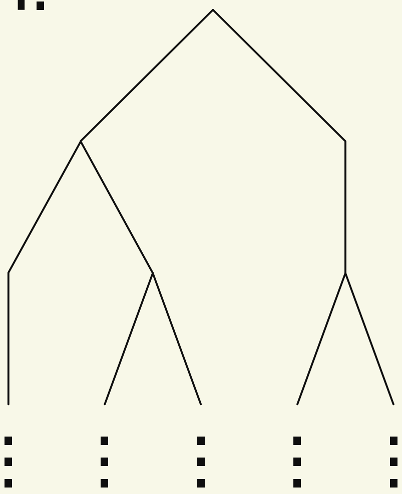
$\therefore$ Every function $f : \mathcal{C} \rightarrow \mathcal{C}$ is continuous.

Axiom of intuitionistic mathematics 2

In the classical mathematics,

König's lemma (KL)

Every infinite binary tree has an infinite path.



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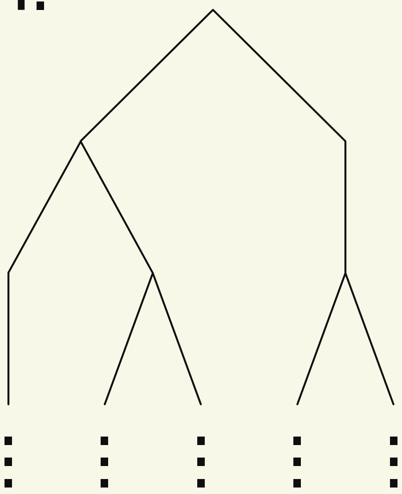

In Brouwerian mathematics,

Brouwer's fan theorem (BFT)

For any binary tree T,
if T has no infinite path, then T is finite.

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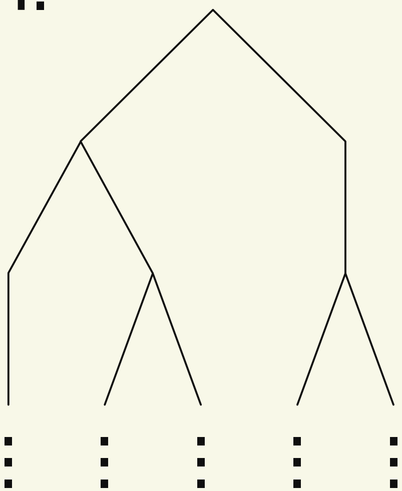

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A intuitionistic counterexample of König's lemma

We have a tree T with paths of any length
but we can prove neither

- T has an infinite path, nor
- T has no infinite path

$\leq$ and $\geq$ in intuitionistic mathematics

Let k_{99} the least n such that

$$\pi = 3.141592\ldots\ldots \quad \overbrace{99999\ldots\ldots9}^{99\text{-length}} \quad \ldots\ldots$$

$\uparrow$

n-th digit

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Let $\rho = \lim_{n\to\infty} a_n$, where

$$a_n = \begin{cases} 0 & \text{if } n < k_{99} \\ -1/k_{99} & \text{if } k_{99} \leq n \text{ and } k_{99} \text{ is even} \\ 1/k_{99} & \text{if } k_{99} \leq n \text{ and } k_{99} \text{ is odd} \end{cases}$$

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If $\rho \leq 0$ (resp. $\rho \geq 0$),
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If $\rho \leq 0$ (resp. $\rho \geq 0$),
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So we do not have $\rho \leq 0 \vee \rho \geq 0$!!

A continuous function on $[0,1]$

In classical mathematics:

Any continuous function $f : [0,1] \to \mathbb{R}$ has minimum value, i.e.,

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$$(\exists v\in\mathbb{R})((\forall y\in[0,1])v\leq f(y))\wedge \\ ((\forall\varepsilon>0)(\exists x\in[0,1])f(x)<v+\varepsilon)$$

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$\longrightarrow$We do not have the maximum value of f

Intuitionistic version of minimax theorem

In classical mathematics:

> For any von Neumann's game, the pair of optimal mixed strategies exists, i.e.,
>
> $$\max_{\sigma \in \mathrm{MS}_I} \min_{\tau \in \mathrm{MS}_{II}} \mathrm{E}(\sigma, \tau) = \min_{\tau \in \mathrm{MS}_{II}} \max_{\sigma \in \mathrm{MS}_I} \mathrm{E}(\sigma, \tau),$$

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In Brouwerian mathematics:

Theorem (Ewaltz)

For any von Neumann's game, the equilibrium point exists in the following sense

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The following game does not have the optimal pair of strategies:

I \ II	str.A	str.B
str.1	0	ρ
str.2	$-\rho$	0

Intuitionistic Blackwell determinacy

In classical mathematics:

> In ZFC, every Borel Blackwell game (i.e., pay-off function is Borel measurable) in $2^{\mathbb{N}}$ is determinate

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Theorem

> Every Blackwell game is determinate.

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But in intuitionistic mathematics, we do not have ordinary determinacy!!

Further problem:

- In intuitionistic mathematics,
  does Blackwell determinacy prove ordinary determinacy of some certain class of games?
  (In classical mathematics, this is partially solved)

Summarize

In intuitionistic mathematics, we have

- Modified version of minimax theorem:

  For any von Neumann's game, the following holds

$$\sup_{\sigma \in \mathrm{MS}_I} \inf_{\tau \in \mathrm{MS}_{II}} \mathrm{E}(\sigma, \tau) = \inf_{\tau \in \mathrm{MS}_{II}} \sup_{\sigma \in \mathrm{MS}_I} \mathrm{E}(\sigma, \tau),$$

- Full Blackwell determinacy in $2^{\mathbb{N}}$